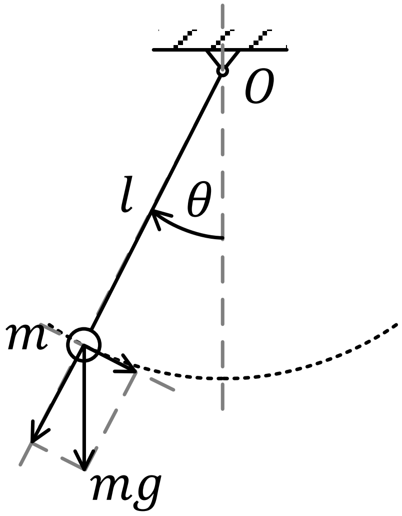
UNIVERZITET U NOVOM SADU
FAKULTET TEHNIČKIH NAUKA
KATEDRA ZA AUTOMATIKU I UPRAVLJANJE SISTEMIMA

# Primeri mehaničkih sistema

Modeliranje i simulacija sistema

Upravljanje, modelovanje i simulacija sistema

# Matematičko klatno

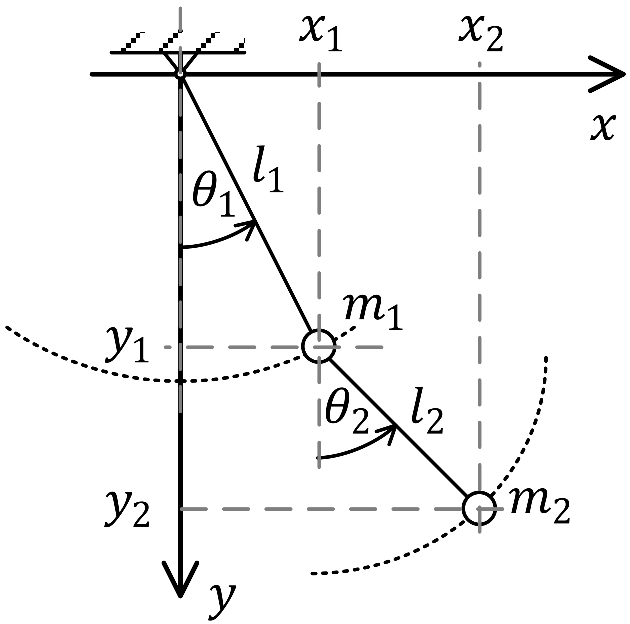



Aksijalne sile – II Njutnov zakon:

$$ml\ddot{\theta} + mg\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

# Dvostepeno matematičko klatno




Lagranževe diferencijalne jednačine drugog reda

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}}\right) - \frac{\partial E_k}{\partial q} + \frac{\partial E_p}{\partial q} = 0$$

$q$ - generalisana koordinata

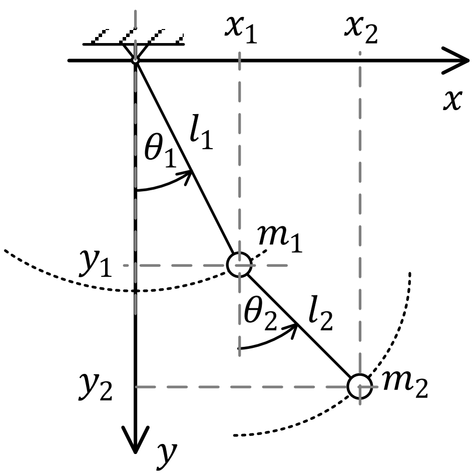
Kinetička energija:

$$E_k = E_{k1} + E_{k2} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

Potencijalna energija:

$$E_p = E_{p1} + E_{p2} = m_1 g(-y_1) + m_2 g(-y_2)$$

# Dvostepeno matematičko klatno (2)




Generalisane koordinate su: $\theta_1$ i $\theta_2$

Smene:

$$x_1 = l_1 \sin\theta_1\,, \qquad y_1 = l_1 \cos\theta_1\,,$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2\,, \quad y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2$$

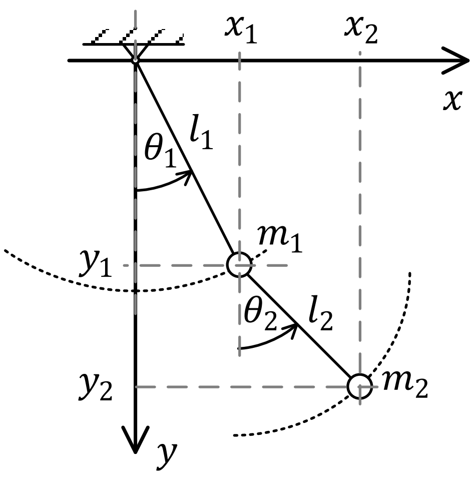
Lagranževe jednačine:

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{\theta}_1}\right) - \frac{\partial E_k}{\partial \theta_1} + \frac{\partial E_p}{\partial \theta_1} = 0,$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{\theta}_2}\right) - \frac{\partial E_k}{\partial \theta_2} + \frac{\partial E_p}{\partial \theta_2} = 0$$

... (traženje izvoda i sređivanje)

# Dvostepeno matematičko klatno (3)




Model sistema:

$$(m_1 + m_2)l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$+ m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g \sin\theta_1 = 0$$

$$m_2 l_1 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)$$
$$+ m_2 g \sin\theta_2 = 0$$

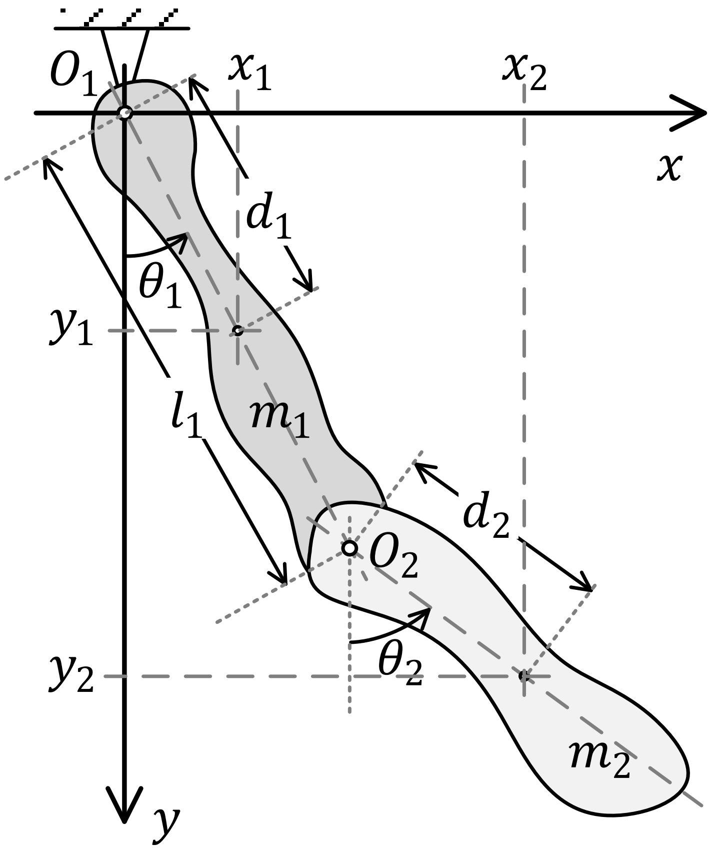
Za male uglove je:

$$\cos(\theta_1 - \theta_2) \approx 1, \sin(\theta_1 - \theta_2) \approx 0,$$
$$\sin\theta_1 \approx \theta_1, \sin\theta_2 \approx \theta_2$$

Model postaje:

$$(m_1 + m_2)l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 + (m_1 + m_2) g\theta_1 = 0$$
$$m_2 l_1 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 + m_2 g \theta_2 = 0.$$

# Fizičko dvospetepno klatno




Kinetička energija:

$$E_k = E_{k1} + E_{k2}$$

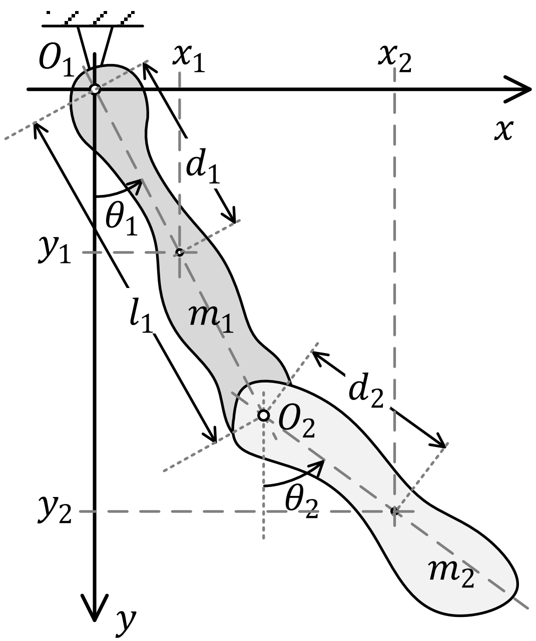
$$E_{k1} = \frac{1}{2} J_1 \dot{\theta}_1^2$$

$$E_{k2} = \frac{1}{2} J_2 \dot{\theta}_2^2 + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

Potencijalna energija:

$$E_p = E_{p1} + E_{p2} = m_1 g(-y_1) + m_2 g(-y_2)$$

# Fizičko dvospetepno klatno (2)




Generalisane koordinate su: $\theta_1$ i $\theta_2$

Smene:

$$x_1 = d_1 \sin\theta_1\,, \qquad y_1 = d_1 \cos\theta_1\,,$$

$$x_2 = l_1 \sin\theta_1 + d_2 \sin\theta_2\,, \quad y_2 = l_1 \cos\theta_1 + d_2 \cos\theta_2$$

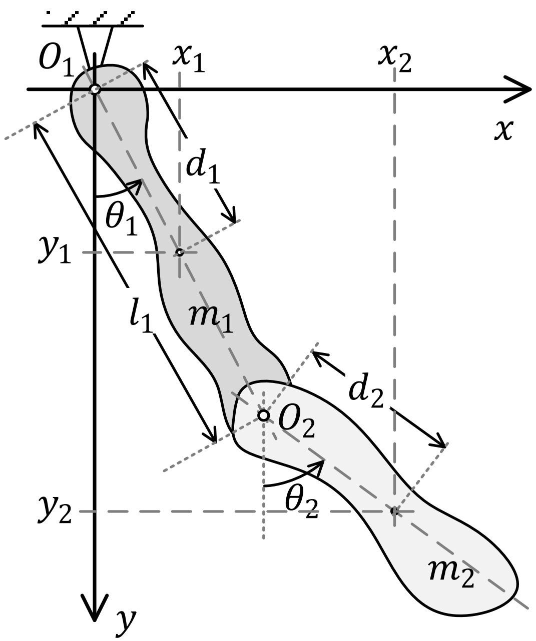
Lagranževe jednačine:

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{\theta}_1}\right) - \frac{\partial E_k}{\partial \theta_1} + \frac{\partial E_p}{\partial \theta_1} = 0,$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{\theta}_2}\right) - \frac{\partial E_k}{\partial \theta_2} + \frac{\partial E_p}{\partial \theta_2} = 0$$

... (traženje izvoda i sređivanje)

# Fizičko dvospetepno klatno (3)




Model sistema:

$$J_1\ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 d_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$- m_2 l_1 d_2 \dot{\theta}_2 \left(\dot{\theta}_1 - \dot{\theta}_2\right) \sin(\theta_1 - \theta_2)$$
$$+ m_2 l_1 d_2 \dot{\theta}_1 \, \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_1 g d_1 sin(\theta_1)$$
$$+ m_2 g l_1 \sin\theta_1 = 0$$

$$J_2\ddot{\theta}_2 + m_2 d_2^2 \ddot{\theta}_2 + m_2 l_1 d_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2)$$
$$- m_2 l_1 d_2 \dot{\theta}_1 \left(\dot{\theta}_1 - \dot{\theta}_2\right) \sin(\theta_1 - \theta_2)$$
$$- m_2 l_1 d_2 \dot{\theta}_1 \, \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g d_2 \sin\theta_2 = 0$$

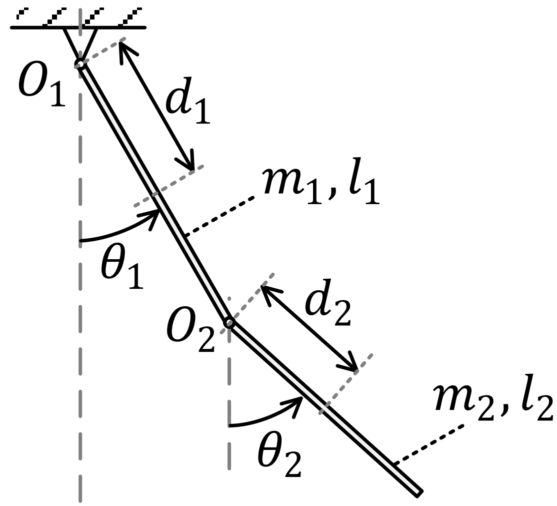
Za male uglove je:

$$\cos(\theta_1 - \theta_2) \approx 1, \sin(\theta_1 - \theta_2) \approx 0,$$
$$\sin\theta_1 \approx \theta_1, \sin\theta_2 \approx \theta_2$$

Model postaje:

$$J_1\ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 d_2 \ddot{\theta}_2 + m_1 g d_1 \theta_1 + m_2 g l_1 \theta_1 = 0$$
$$J_2\ddot{\theta}_2 + m_2 d_2^2 \ddot{\theta}_2 + m_2 l_1 d_2 \ddot{\theta}_1 + m_2 g d_2 \theta_2 = 0$$

# Dvostepeno fizičko klatno od štapova




$l_1 = l_2 = l,\ d_1 = d_2 = \frac{l}{2},\ m_1 = m_2 = m,$

$J_1 = \frac{1}{3} m l^2,\ J_2 = \frac{1}{12} m l^2$

Model

$$\ddot{\theta}_1 + \frac{3}{8}\left[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)\right] + \frac{9g}{8l} \sin\theta_1 = 0$$

$$\ddot{\theta}_2 + \frac{3}{2}\left[\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)\right] + \frac{3g}{2l} \sin\theta_2 = 0$$

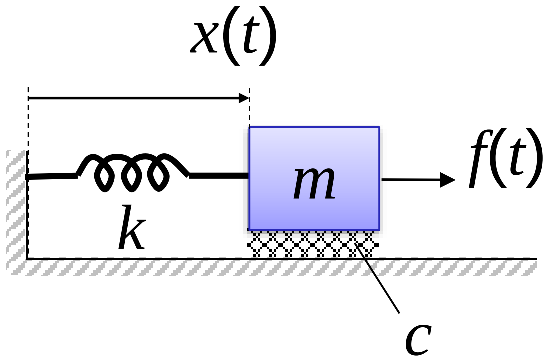
Mali uglovi

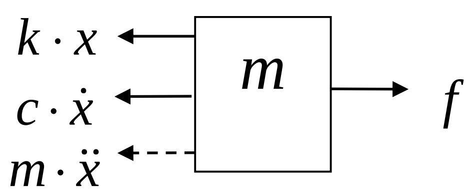
$$\ddot{\theta}_1 + \frac{3}{8}\ddot{\theta}_2 + \frac{9g}{8l}\,\theta_1 = 0$$

$$\ddot{\theta}_2 + \frac{3}{2}\ddot{\theta}_1 + \frac{3g}{2l}\theta_2 = 0$$

# Amortizer

# Matematički model




Na osnovu Dalamberovog zakona

$$\sum_i f_i = 0$$

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

# Nema pobudne sile

$$m\ddot{x} + c\dot{x} + kx = 0, \quad x(0) = x_0, \quad \dot{x}(0) = v_0$$ ← homogena jednačina

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0, \quad \omega_0 = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{c}{2\sqrt{mk}}$$

Karakteristična jednačina:

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

$$s_{1,2} = -\zeta\omega_0 \mp \omega_0\sqrt{\zeta^2 - 1}$$

$$D = \zeta^2 - 1$$

$D > 0$, tj. $\zeta > 1$ daje aperiodičan odziv

$D = 0$, tj. $\zeta = 1$ daje kritično-aperiodičan odziv

$D < 0$, tj. $0 \leq \zeta < 1$ daje periodičan odziv

# Aperiodičan odziv: $\zeta > 1$
# Analitičko rešenje

$$x(t) = q_1 e^{s_1 t} + q_2 e^{s_2 t}$$

$q_1$ i $q_2$ se određuju na osnovu početnih vrednosti $x_0$ i $v_0$

$$x_0 = q_1 e^{s_1 0} + q_2 e^{s_2 0} = q_1 + q_2$$

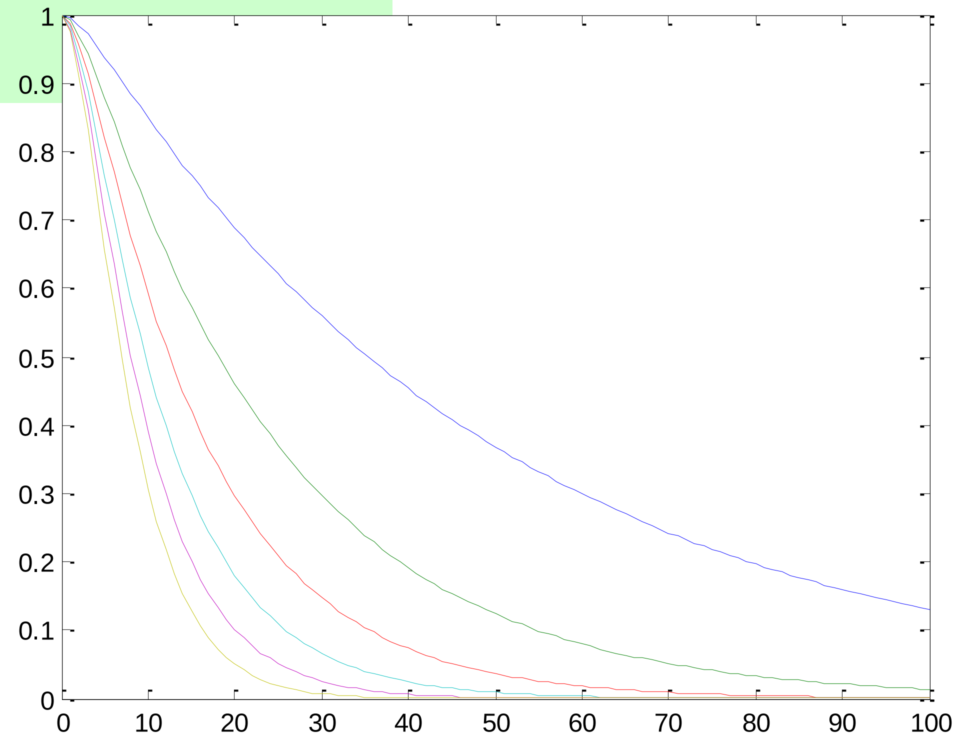
$$v_0 = \frac{d}{dt} x(t) \Big|_{t=0} = q_1 s_1 + q_2 s_2$$

$$q_1 = x_0 + \frac{s_1 x_0 - v_0}{s_1 - s_2}$$

$$q_2 = -\frac{s_1 x_0 - v_0}{s_1 - s_2}$$

# Aperiodičan odziv: $\zeta > 1$
# Simulacija

```
m=10; c=5;
X = []; ksi = []; w0=[];
for k = [0.1 0.2 0.3 0.4 0.5 0.6]
    ksi = [ksi c/2/sqrt(m*k)];
    w0 = [w0 sqrt(k/m)];
    [t,x]=ode45(@oscil2,[0:100],[1;0],[],m,c,k);
    X = [X x(:,1)];
end
plot(t,X)
```




```
function xp = oscil2(t,x,m,c,k)
xp = [ x(2)
      -c/m*x(2)-k/m*x(1) ];
```

# Aperiodičan odziv: $\zeta > 1$

```
>> w0
w0 =
    0.1000    0.1414    0.1732    0.2000    0.2236    0.2449
>> ksi
ksi =
    2.5000    1.7678    1.4434    1.2500    1.1180    1.0206
>> D = ksi.^2-1
D =
    5.2500    2.1250    1.0833    0.5625    0.2500    0.0417
>> s1 = -ksi.*w0 + w0.*sqrt(ksi.^2-1)
s1 =
   -0.0209   -0.0438   -0.0697   -0.1000   -0.1382   -0.2000
>> s2 = -ksi.*w0 - w0.*sqrt(ksi.^2-1)
s2 =
   -0.4791   -0.4562   -0.4303   -0.4000   -0.3618   -0.3000
```

# Kritično-aperiodičan odziv: $\zeta = 1$ Analitičko rešenje

$$x(t) = (q_1 + q_2 t)e^{-\omega_0 t}$$

$q_1$ i $q_2$ se određuju na osnovu početnih vrednosti $x_0$ i $v_0$

$$x_0 = q_1$$

$$v_0 = \frac{d}{dt}x(t)\bigg|_{t=0} = q_2 - q_1\omega_0$$

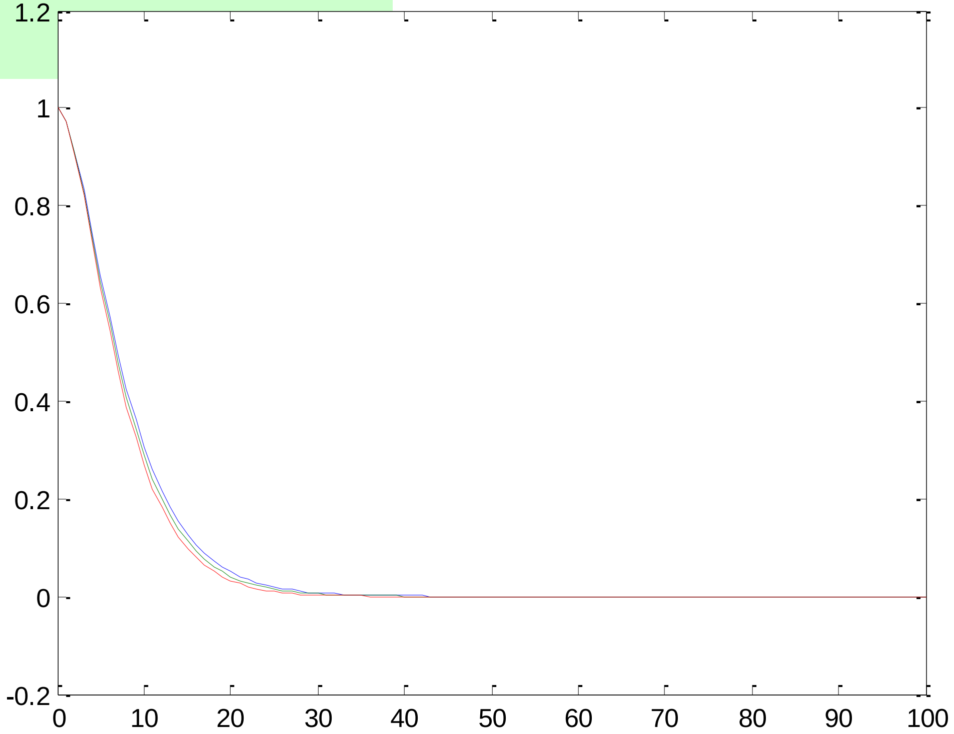
$\Rightarrow$

$$q_1 = x_0$$

$$q_2 = v_0 + \omega_0 x_0$$

# Kritično-aperiodičan odziv: $\zeta = 1$ Simulacija

```
m=10; c=5;
X = []; ksi = []; w0=[];
for k = [0.6 0.625 0.65]
    ksi = [ksi c/2/sqrt(m*k)];
    w0 = [w0 sqrt(k/m)];
    [t,x]=ode45(@oscil2,[0:100],[1;0],[],m,c,k);
    X = [X x(:,1)];
end
plot(t,X)
```




```
>> min(X)
ans =
  1.0e-006 *
    0.0130    0.0024   -0.1793
```

# Kritično-aperiodičan odziv: $\zeta = 1$

```
>> ksi
ksi =
    1.0206    1.0000    0.9806
>> w0
w0 =
    0.2449    0.2500    0.2550
>> D = ksi.^2-1
D =
    0.0417         0   -0.0385
>> s1=-ksi.*w0+w0.*sqrt(ksi.^2-1)
s1 =
  -0.2000            -0.2500            -0.2500 + 0.0500i
>> s2=-ksi.*w0-w0.*sqrt(ksi.^2-1)
s2 =
  -0.3000            -0.2500            -0.2500 - 0.0500i
```

# Periodičan odziv: $0 \le \zeta < 1$
# Analitičko rešenje

$$x(t) = e^{-\zeta\omega_0 t}(q_1 \cos\omega_d t + q_2 \sin\omega_d t), \quad \omega_d = \omega_0\sqrt{1-\zeta^2}$$

$q_1$ i $q_2$ se određuju na osnovu početnih vrednosti $x_0$ i $v_0$

$$x_0 = q_1$$

$$v_0 = \frac{d}{dt}x(t)\Big|_{t=0} = -\zeta\omega_0 q_1 + \omega_d q_2$$

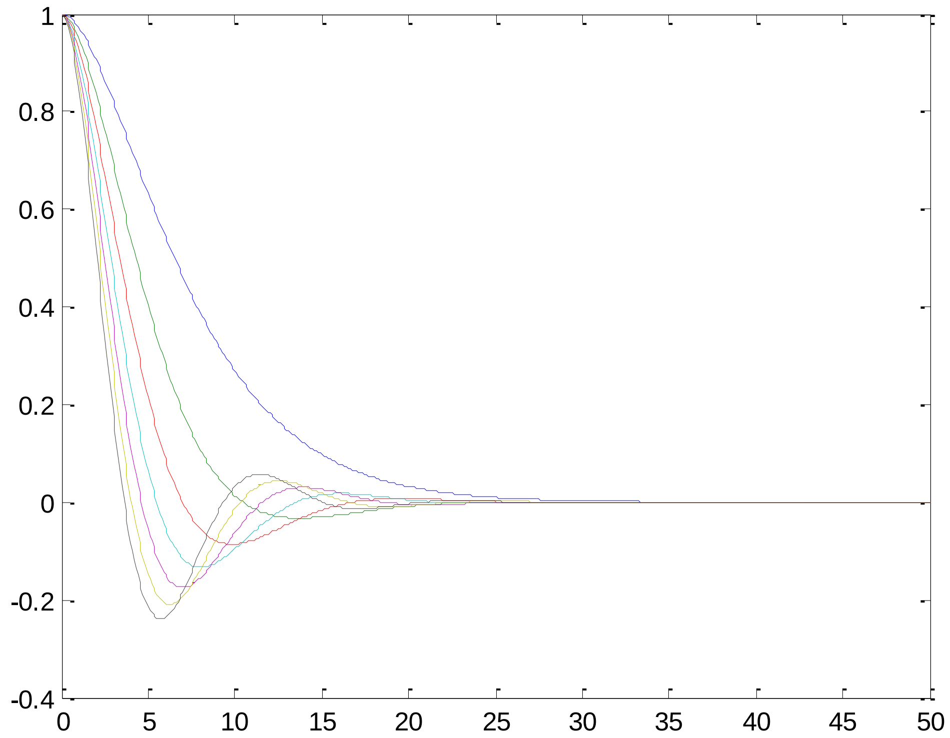
$\Rightarrow$

$$q_1 = x_0$$

$$q_2 = \frac{1}{\omega_d}(v_0 + \zeta\omega_0 x_0)$$

# Periodičan odziv: $0 \leq \zeta < 1$ Simulacija

```
m=10; c=5;
X = []; ksi = []; w0=[];
for k = 0.65:0.5:4
ksi = [ksi c/2/sqrt(m*k)];
    w0 = [w0 sqrt(k/m)];
    [t,x]=ode45(@oscil2,...
        [0:50],[1;0],[],m,c,k);
    X = [X x(:,1)];
end
plot(t,X)
```




```
>> min(X)
ans =
    0.0000    -0.0325    -0.0860    -0.1338    -0.1745    -0.2094    -0.2397
```

# Periodičan odziv: $0 \leq \zeta < 1$
# Simulacija

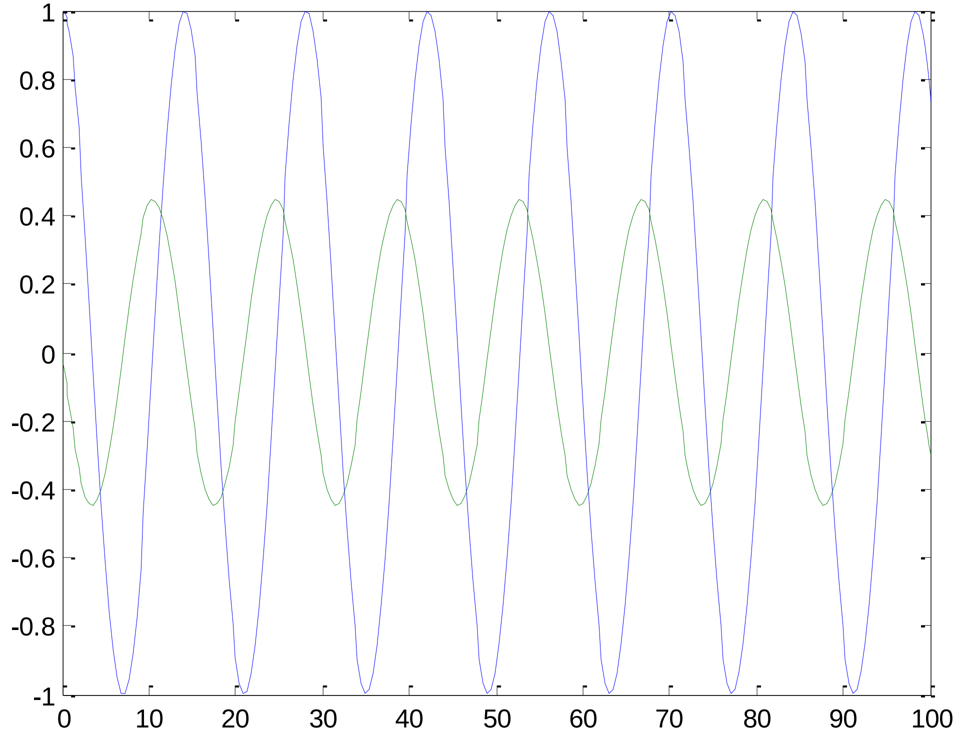
```
>> ksi
ksi =
    0.9806    0.7372    0.6155    0.5392    0.4856    0.4454    0.4138
>> w0
w0 =
    0.2550    0.3391    0.4062    0.4637    0.5148    0.5612    0.6042
>> D = ksi.^2-1
D =
   -0.0385   -0.4565   -0.6212   -0.7093   -0.7642   -0.8016   -0.8288
>> s1=-ksi.*w0+w0.*sqrt(ksi.^2-1)
s1 =
  -0.2500 + 0.0500i  -0.2500 + 0.2291i  -0.2500 + 0.3202i   ...
  -0.2500 + 0.3905i  -0.2500 + 0.4500i  -0.2500 + 0.5025i   ...
  -0.2500 + 0.5500i
>> s2=-ksi.*w0-w0.*sqrt(ksi.^2-1)
s2 =
  -0.2500 - 0.0500i  -0.2500 - 0.2291i  -0.2500 - 0.3202i   ...
  -0.2500 - 0.3905i  -0.2500 - 0.4500i  -0.2500 - 0.5025i   ...
  -0.2500 - 0.5500i
```

# Neprigušene oscilacije

```
m=10; c=0; k=2;
[t,x]=ode45(@oscil2,[0 100],[1;0],[],m,c,k);
plot(t,x)

ksi = c/2/sqrt(m*k)
w0 = sqrt(k/m)
wd = w0 * sqrt(1-ksi^2)
T = 2*pi/wd      % perioda oscilacija
```

```
ksi =
     0
w0 =
    0.4472
wd =
    0.4472
T =
   14.0496
```

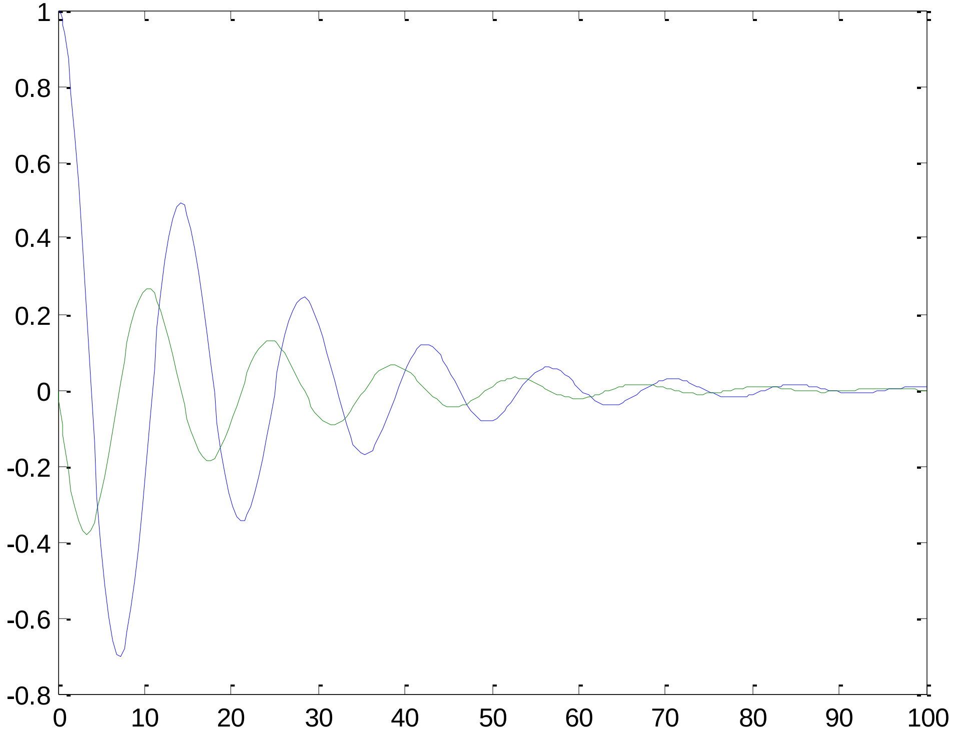


# Prigušene oscilacije

```
m=10; c=1; k=2;
[t,x]=ode45(@oscil2,[0 100],[1;0],[],m,c,k);
plot(t,x)

ksi = c/2/sqrt(m*k)
w0 = sqrt(k/m)
wd = w0 * sqrt(1-ksi^2)
T = 2*pi/wd
```

```
ksi =
    0.1118
w0 =
    0.4472
wd =
    0.4444
T =
   14.1383
```



# Pobuda nije nula

$m\ddot{x}+c\dot{x}+kx=F\cos(\omega t)$ problem

$x(t)=x_p(t)+x_h(t)$ oblik rešenja

partikularni deo

$$x_p(t)=A\sin(\omega t)+B\cos(\omega t),\quad A=\frac{\omega cF}{(k-\omega^2 m)^2+(\omega c)^2},\quad B=\frac{A(k-\omega^2 m)}{\omega c}$$

homogeni deo

$m\ddot{x}+c\dot{x}+kx=0$

$$\ddot{x}+2b\dot{x}+a^2x=0,\quad a=\sqrt{\frac{k}{m}},\quad b=\frac{c}{2m}$$

karakteristična jednačina

$s^2+2bs+a^2=0$

$$s_{1,2}=-b\mp\sqrt{b^2-a^2}=-b\mp\sqrt{D}$$

$D$>0 daje aperiodičan odziv $x_h(t)=q_1e^{s_1t}+q_2e^{s_2t}$

$D$=0 daje kritično-aperiodičan odziv $x_h(t)=q_1e^{-bt}+q_2te^{-bt}$

$D$<0 daje periodičan odziv $x_h(t)=e^{-bt}(q_1\cos at+q_2\sin at), a=\sqrt{-D}$

# Rešenje - konkretno (2)

*Zadato:* $m$=10kg, $c$=5kg/s, $\omega$=2rad/s i

U početnom trenutku teg se nalazi na rastojanju 1m od ravnotežnog položaja i nema brzinu.

- a) $k$=0.5, $F$=0;
- b) $k$=10/16, $F$=0;
- c) $k$=4, $F$=0;
- d) $k$=0.5, $F$=10N;
- e) $k$=10/16, $F$=10;
- f) $k$=4, $F$=10

$$m\ddot{x} + c\dot{x} + kx = F\cos(\omega t)$$

Kretanje sistema posmatrati tokom prvih 40 sekundi.

Smene:

$$x_1(t) = x(t) \qquad \dot{x}_1(t) = x_2(t)$$

$$x_2(t) = \dot{x}(t) \qquad \dot{x}_2(t) = \frac{F}{m}\cos(\omega t) - \frac{c}{m}x_2(t) - \frac{k}{m}x_1(t)$$

# Matlab rešenje

```
function xp = oscil(t,x,F,m,w,c,k)
xp = [ x(2); F/m*cos(w*t)-c/m*x(2)-k/m*x(1) ];
```

```
function testosc(m,c,k,F,w,Tk)
x0 = [1; 0];                        % pocetno stanje, x(0)=1, v=0
t = 0:Tk/1000:Tk;                   % vremenska osa
b = c/2/m;, a = sqrt(k/m);          % mx^2+cx+k==x^2+2bx+a^2
D = b^2-a^2;
disp(['D=', num2str(D)])
A = w*c*F/((k-w^2*m)^2+(w*c)^2);
B = A/w/c*(k-w^2*m);
xp = A*sin(w*t) + B*cos(w*t);    % partikularno resenje
if D > 0
  disp('prosti, razliciti polovi')
  s1 = -b+sqrt(D);
  s2 = -b-sqrt(D);
  q = [1 1; s1 s2] \ (x0-[B;w*A]);
  xh = q(1)*exp(s1*t)+q(2)*exp(s2*t);
elseif D == 0
  disp('dvostruk pol')
  q = [1 0; -b 1] \ (x0-[B;w*A]);
  xh = (q(1)+q(2)*t).*exp(-b*t);                                 ...
```

```
else
  disp('konjugovano-kompleksni polovi')
  a = sqrt(-D);
  q = [1 0; -b a] \ (x0-[B;w*A]);
  xh = exp(-b*t).*(q(1)*cos(a*t)+q(2)*sin(a*t));
end
x = xp + xh;                        % part. + homogeno res.

[ts,xs] = ode23(@oscil,[0,Tk],x0,[], F,m,w,c,k);
figure(1)
plot(t,x,'r',ts,xs(:,1),'b.-'), title('Tacno i sim. resenje')
% crtanje razlike
xl=interp1(t,x,ts);                 % postavi resenja u isto vreme
figure(2)
plot(ts,[xs(:,1)-xl]), title('razlike')
```

```
disp('Priguseno oscilovanje')
%---------m--c------k---F--w--Tk---
testosc( 10, 5,   0.5,  0, 2, 40 ), pause    % a
testosc( 10, 5, 10/16,  0, 2, 40 ), pause    % b
testosc( 10, 5,     4,  0, 2, 40 ), pause    % c
disp('Prinudno oscilovanje')
testosc( 10, 5,   0.5, 10, 2, 40 ), pause    % d
testosc( 10, 5, 10/16, 10, 2, 40 ), pause    % e
testosc( 10, 5,     4, 10, 2, 40 )           % f
```

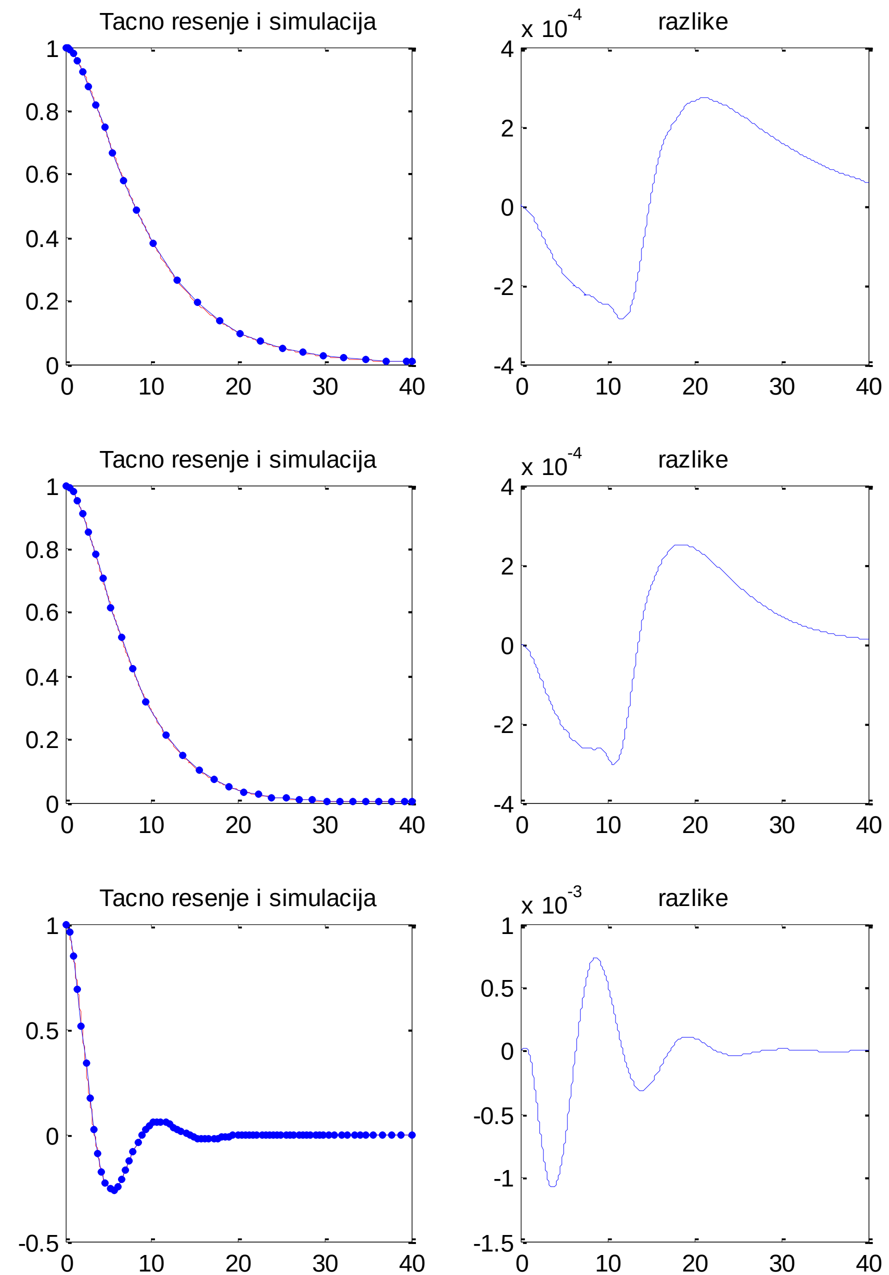
# Rezultati

**Prigušeno oscilovanje**

- D=0.0125
  prosti, različiti polovi

- D=0
  dvostruk pol

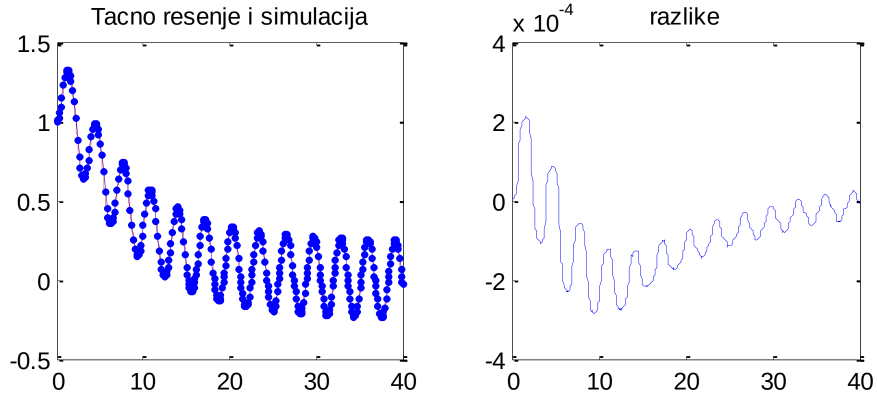
- D=-0.3375
  konjugovano-kompleksni
  polovi



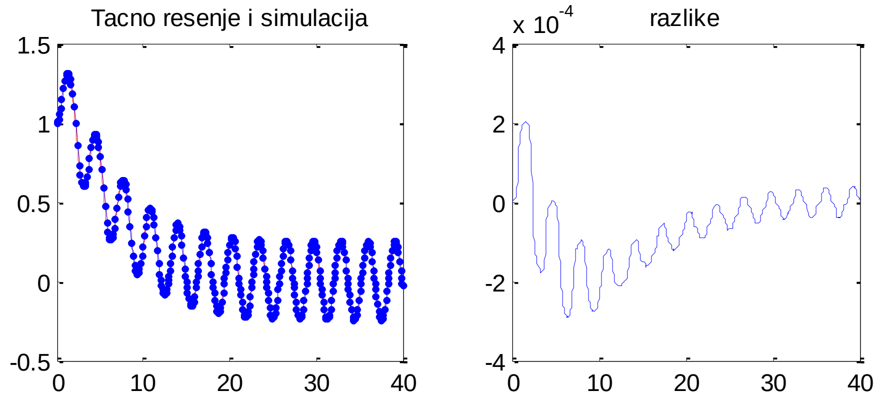
# Rezultati (2)

**Prinudno oscilovanje**

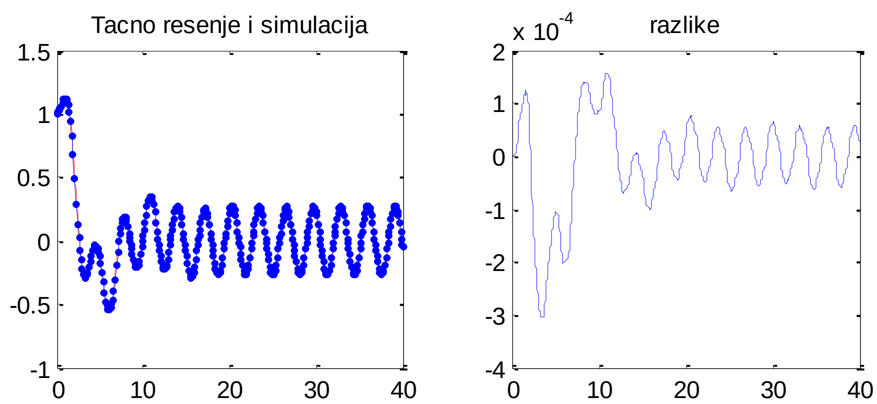
- D=0.0125
  prosti, različiti polovi




- D=0
  dvostruk pol




- D=-0.3375
  konjugovano-kompleksni polovi



# Odziv usled početne vrednosti i pobude

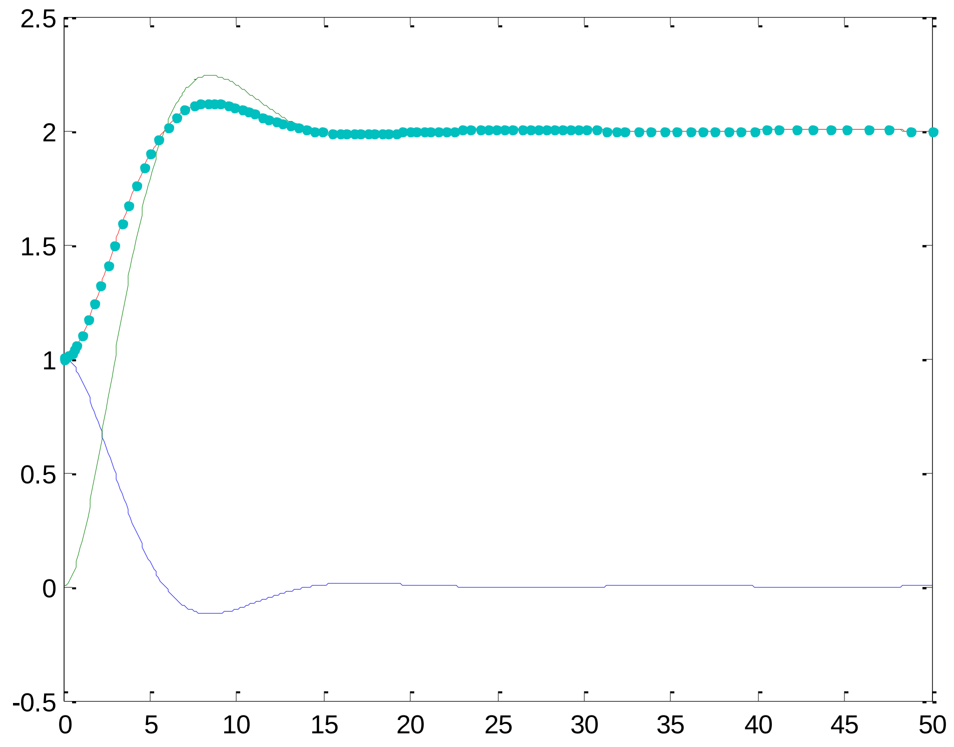
```
m=10; c=5; k=2; tk=50; tout=0:0.1:tk;
[t,x]=ode45(@oscil3,tout,[1;0],[],m,c,k,0,0,0); x1=x(:,1);
[t,x]=ode45(@oscil3,tout,[0;0],[],m,c,k,4,0,0); x2=x(:,1);
x3 = x1 + x2;
[tr,xr]=ode45(@oscil3,[0 tk],[1;0],[],m,c,k,4,0,0);
plot(t,[x1 x2 x3], tr,xr(:,1),'.')
```

```
function xp = oscil3(t,x,m,c,k,f0,f,w)
xp = [ x(2); (f0+f*cos(w*t))/m-c/m*x(2)-k/m*x(1) ];
```

$$m\ddot{x} + c\dot{x} + kx = F_0 = 4$$

$$x(0) = x_0 = 1$$

$$\dot{x}(0) = v_0 = 0$$



# Odziv usled složene pobude

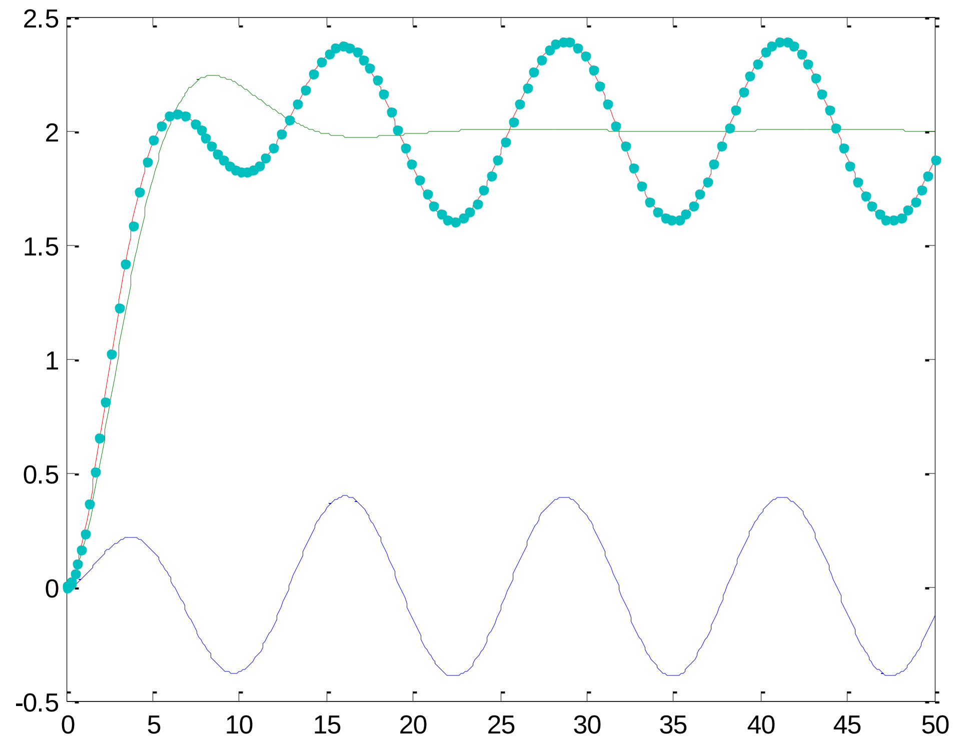
```
m=10; c=5; k=2; tk=50; tout=0:0.1:tk;
[t,x]=ode45(@oscil3,tout,[0;0],[],m,c,k,0,1,0.5); x1=x(:,1);
[t,x]=ode45(@oscil3,tout,[0;0],[],m,c,k,4,0,0);   x2=x(:,1);
x3 = x1 + x2;
[tr,xr]=ode45(@oscil3,[0 tk],[0;0],[],m,c,k,4,1,0.5);
plot(t,[x1 x2 x3], tr,xr(:,1),'.')
```

```
function xp = oscil3(t,x,m,c,k,f0,f,w)
xp = [ x(2); (f0+f*cos(w*t))/m-c/m*x(2)-k/m*x(1) ];
```

$$m\ddot{x} + c\dot{x} + kx = F_0 + F\cos(\omega t)$$

$$x(0) = 0$$

$$\dot{x}(0) = 0$$



# Nelinearan model (1)

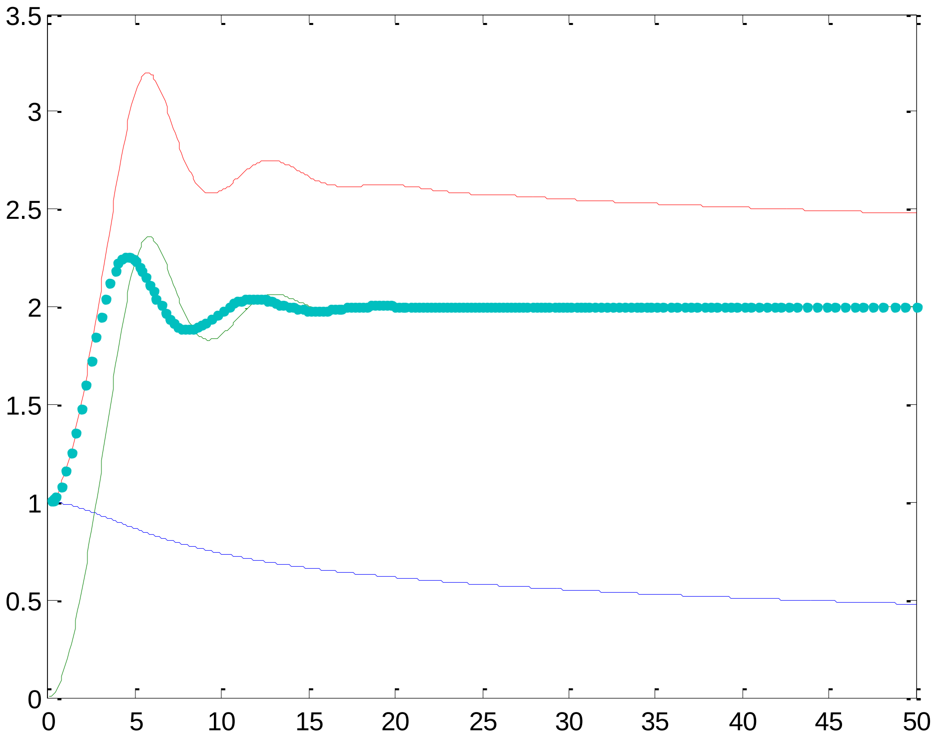
```
function xp = oscil4(t,x,m,c,k,f0,f,w)
k1=k/8*x(1)^3;
xp = [ x(2); (f0+f*cos(w*t))/m-c/m*x(2)-k1/m ];
```

```
m=10; c=5; k=2;
tk=50; tout=0:0.1:tk;
[t,x]=ode45(@oscil4,tout,...
        [1;0],[],m,c,k,0,0,0);
x1=x(:,1);
[t,x]=ode45(@oscil4,tout,...
        [0;0],[],m,c,k,4,0,0);
x2=x(:,1);
x3 = x1 + x2;
[tr,xr]=ode45(@oscil4,[0 tk],...
        [1;0],[],m,c,k,4,0,0);
plot(t,[x1 x2 x3],tr,xr(:,1),'.')
```

$$m\ddot{x} + c\dot{x} + \frac{k}{8}x^3 = F_0 = 4$$

$$x(0) = x_0 = 1$$

$$\dot{x}(0) = v_0 = 0$$



# Nelinearan model (2)

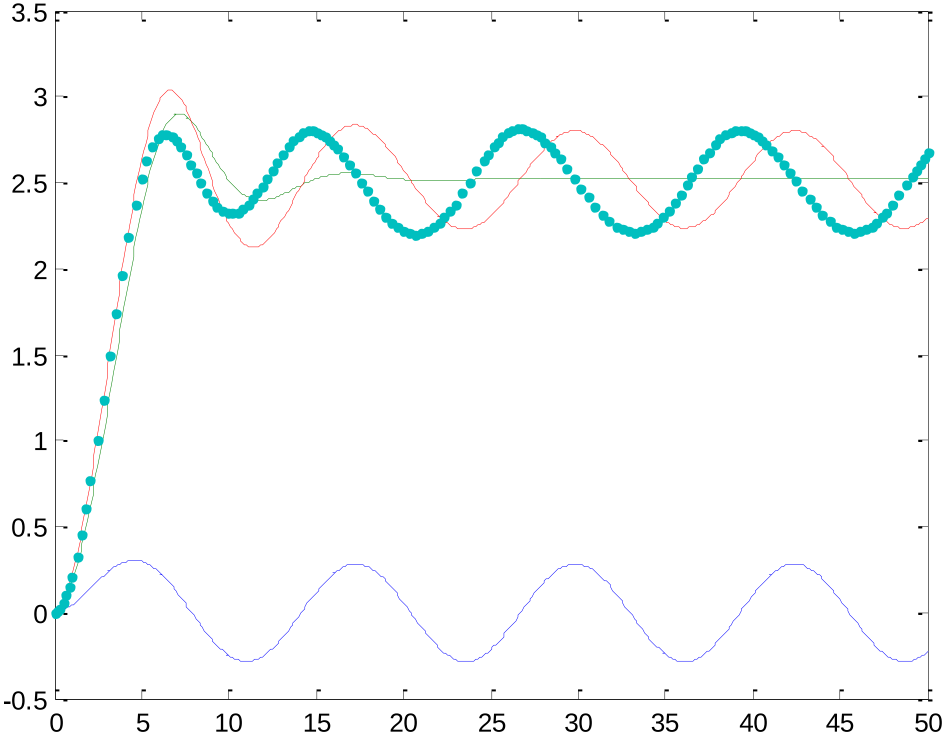
```
function xp = oscil4(t,x,m,c,k,f0,f,w)
k1=k*x(1)^3/8;
xp = [ x(2); (f0+f*cos(w*t))/m-c/m*x(2)-k1/m*x(1) ];
```

```
m=10; c=5; k=2;
tk=50; tout=0:0.1:tk;
[t,x]=ode45(@oscil4,tout,...
        [0;0],[],m,c,k,0,1,0.5);
x1=x(:,1);
[t,x]=ode45(@oscil4,tout,...
        [0;0],[],m,c,k,4,0,0);
x2=x(:,1);
x3 = x1 + x2;
[tr,xr]=ode45(@oscil4,[0 tk],...
        [0;0],[],m,c,k,4,1,0.5);
plot(t,[x1 x2 x3],tr,xr(:,1),'.')
```

$$m\ddot{x} + c\dot{x} + \frac{k}{8}x^3 = F_0 + F\cos(\omega t)$$

$$x(0) = 0$$

$$\dot{x}(0) = 0$$

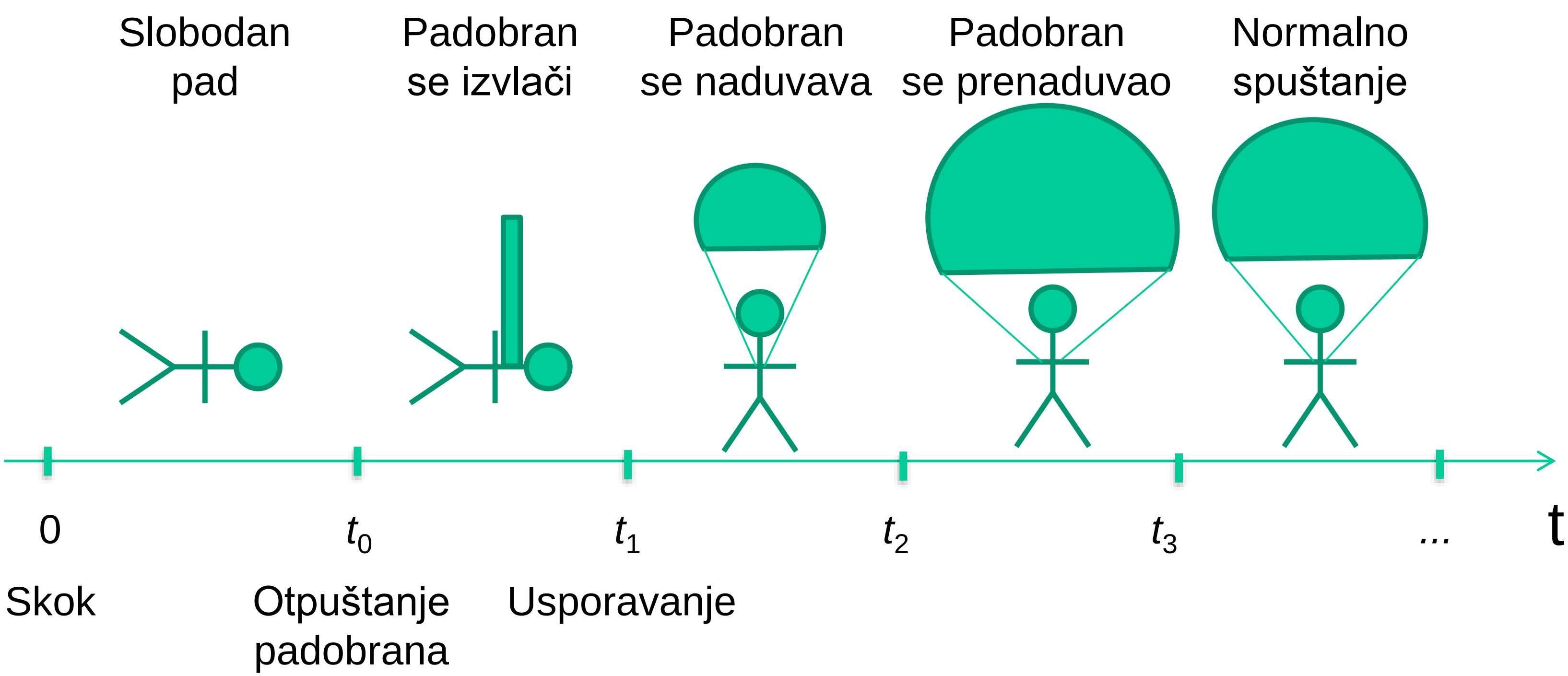


# Padobranac

# Model padobranca

$$m\frac{dv}{dt}+kv^2=mg,\quad v(0)=0$$

$$k=\tfrac{1}{2}\rho(C_sP_s+C_pP_p)$$



# Parametri modela

| $a_1$ | $b_0$ | $b_1$ | $h$ | $l$ | $m$ | $t_0$ | $t_1$ | $t_2$ | $t_3$ |
|---|---|---|---|---|---|---|---|---|---|
| 43.8 $m^2$ | 0.5 $m^2$ | 0.1 $m^2$ | 1.78 m | 8.96 m | 97.2 kg | 10 s | 10.5 s | 11,5 s | 13.2 s |
| Površina padobrana | Površina padobranca horizontalna | Površina padobranca vertikalna | Visina padobranca | Dužina kanapa | Težina padobranca | Vreme 0 | Vreme 1 | Vreme 2 | Vreme 3 |

$$k = \tfrac{1}{2}\rho \begin{cases} 1.95b_0, & t \le t_0 \\ 1.95b_0 + 0.35b_1 l\,\frac{t-t_0}{t_1-t_0}, & t_0 < t \le t_1 \\ 0.35b_1 h + 1.33A_1(t), & t_1 < t \le t_2 \\ 0.35b_1 h + 1.33A_2(t), & t_2 < t \le t_3 \\ 0.35b_1 h + 1.33a_1, & t_3 < t \end{cases}$$

$$A_1(t) = \alpha_0 e^{\beta_0 \frac{t-t_1}{t_2-t_1}}$$

$$A_2(t) = a_1\left(1 + \beta_1 \sin\left(\pi \frac{t-t_2}{t_3-t_2}\right)\right)$$

$$\alpha_0 = \frac{1.95b_0 + 0.35b_1(l-h)}{1.33}$$

$$\beta_0 = \ln\frac{a_1}{\alpha_0}$$

$$\beta_1 = 0.15$$

# Simulacija padobranca

```
function vprime=f(t,v, a_1,b_0,b_1,h,l,m,t_0,t_1,t_2,t_3,g,beta_0,beta_1,alpha_0)
k = kfunction(t,a_1,b_0,b_1,h,l,t_0,t_1,t_2,t_3,beta_0,beta_1,alpha_0);
vprime=g-k*v.^2/m;

function k = kfunction(t,a_1,b_0,b_1,h,l,t_0,t_1,t_2,t_3,beta_0,beta_1,alpha_0)
if t<=t_0
    k = 0.5*(1.95*b_0);
elseif t_0<t & t<=t_1
    k = 0.5*(1.95*b_0+0.35*b_1*l*((t-t_0)/(t_1-t_0)));
elseif t_1<t & t<=t_2
    A_1=alpha_0*exp(beta_0*(t-t_1)/(t_2-t_1));
    k = 0.5*(0.35*b_1*h+1.33*A_1);
elseif t_2<t & t<=t_3
    A_2=a_1*(1+beta_1*sin(pi*(t-t_2)/(t_3-t_2)));
    k = 0.5*(0.35*b_1*h+1.33*A_2);
else
    k = 0.5*(0.35*b_1*h+1.33*a_1);
end
```

# Simulacija padobranca (2)

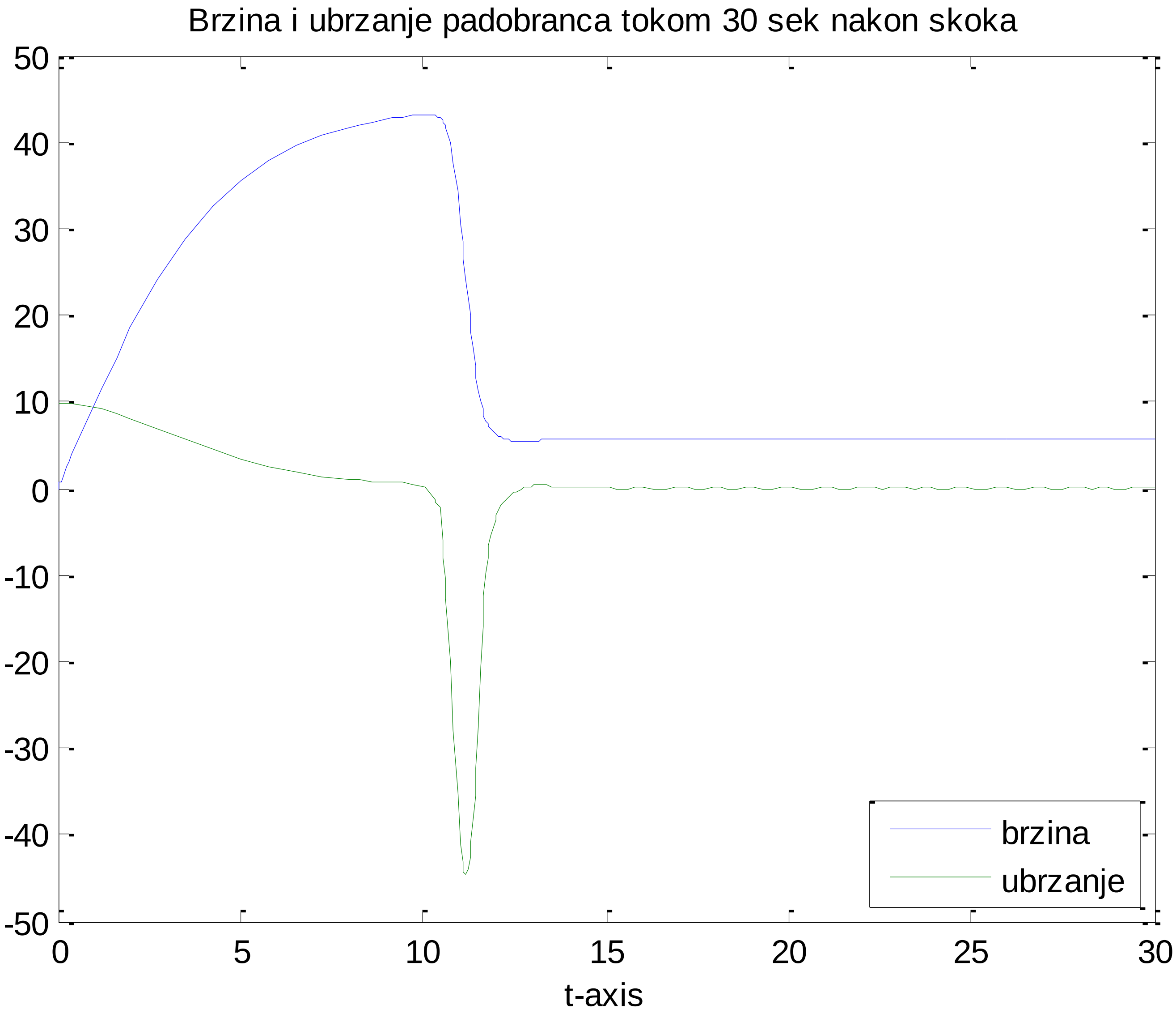
```
a_1=43.8;
b_0=0.5; b_1=0.1;
h=1.778;
l=8.96;
m=97.2;
t_0=10; t_1=10.5; t_2=11.5; t_3=13.2;
g=9.81;
beta_1=0.15;
alpha_0=1.95*b_0+0.35*b_1*(l-h)/1.33;
beta_0=log(a_1/alpha_0);

tspan=[0,30];
init=0;
[t,v]=ode45(@f,tspan,init,[],...
         a_1,b_0,b_1,h,l,m,t_0,t_1,t_2,t_3,g,beta_0,beta_1,alpha_0);

% racunanje ubrzanja
for i=1:length(t)
    k=kfunction(t(i),a_1,b_0,b_1,h,l,t_0,t_1,t_2,t_3,beta_0,beta_1,alpha_0);
    a(i)=g-k*v(i).^2/m;
end

plot(t,v,t,a)
legend('brzina','ubrzanje',4), xlabel('t-axis')
title('Brzina i ubrzanje padobranca tokom 30 sek nakon skoka')
```

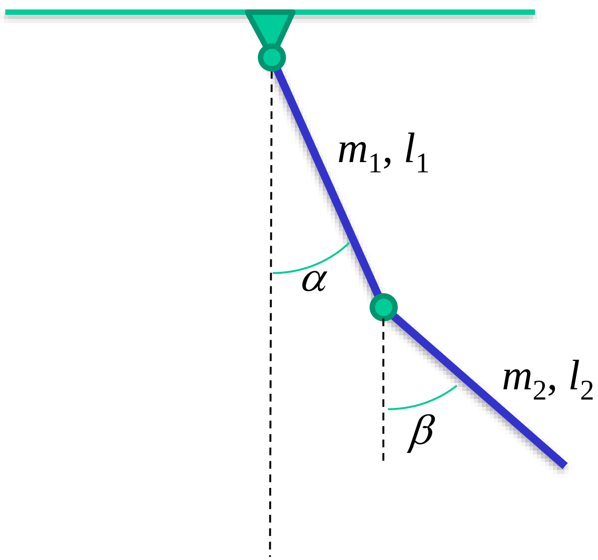
# Rezultat simulacije padobranca



# Fizičko klatno

# Dvostepeno fizičko klatno

Posmatra se dvostepeno klatno sačinjeno od dva štapa masa $m_1$ i $m_2$ i dužina $l_1$ i $l_2$, respektivno. Veze među štapovima su zglobne, bez trenja. Sistem osciluje u vertikalnoj ravni homogenog polja Zemljine teže. Simurati kretanje ovog sistema kada se sistem izvede iz ravnoteže.

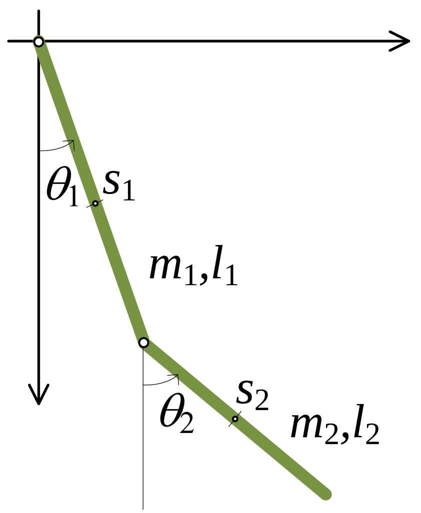


# Matematički model

- Posmatraju se generalisane koordinate: uglovi $\theta_1$ i $\theta_2$ koje klatna zaklapaju sa vertikalom
- Napišu se Lagranževe jednačine druge vrste

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{\theta}_i} - \frac{\partial E_k}{\partial \theta_i} + \frac{\partial E_p}{\partial \theta_i} = 0, \quad i = 1,2$$

gde su: $E_k$ kinetička, a $E_p$ potencijalna energija sistema



# Matematički model (2)

$$a\ddot{\theta}_1 + b\cos(\theta_1-\theta_2)\ddot{\theta}_2 + b\sin(\theta_1-\theta_2)\dot{\theta}_2^2 + e\sin\theta_1 = 0$$

$$f\ddot{\theta}_2 + b\cos(\theta_1-\theta_2)\ddot{\theta}_1 - b\sin(\theta_1-\theta_2)\dot{\theta}_1^2 + k\sin\theta_2 = 0$$

$$a = (\tfrac{1}{3}m_1 + m_2)l_1^2, \quad b = \tfrac{1}{2}m_2 l_1 l_2, \quad e = g(\tfrac{1}{2}m_1 + m_2)l_1^2, \quad f = \tfrac{1}{3}m_2 l_2^2, \quad k = g\tfrac{1}{2}m_2 l_2$$

---

$$x_1 = \theta_1, \quad x_2 = \dot{\theta}_1, \quad x_3 = \theta_2, \quad x_4 = \dot{\theta}_3$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{-bf\sin(x_1-x_3)x_4^2 - ef\sin x_1 - b^2\sin(x_1-x_3)\cos(x_1-x_3)x_2^2 + bk\cos(x_1-x_3)\sin x_3}{af - b^2\cos^2(x_1-x_3)}$$

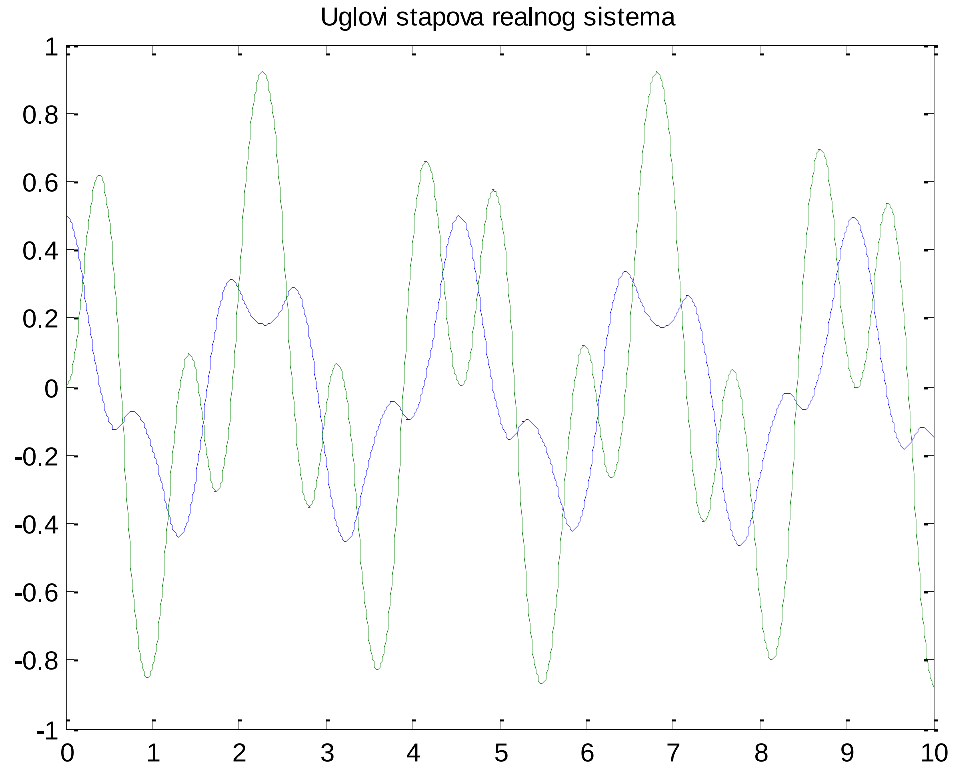
$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{b^2\sin(x_1-x_3)\cos(x_1-x_3)x_2^2 + eb\cos(x_1-x_3)\sin x_1 + ab\sin(x_1-x_3)x_2^2 - ak\sin x_3}{af - b^2\cos^2(x_1-x_3)}$$

# Simulacija dvostepenog klatna

```
g=9.81;
m1=1; m2=1;
l1=1; l2=0.8;
a=(m1+3*m2)*l1^2/3;
b=m2*l1*l2/2;
f=m2*l2^2/3;
e=g*l1*(m1+2*m2)/2;
k=l2*m2*g/2;

x0 = [0.5; 0; 0; 0];
tout = 0:0.01:10;
[t,fi]=ode45(@klatno,tout,x0,...
         [],a,b,e,f,k);
fi=fi(:,[1 3]);
plot(t,fi)
title('Uglovi... ')
```




```
function xp=klatno(t,x,a,b,e,f,k)
c=cos(x(1)-x(3)); s=sin(x(1)-x(3)); im=a*f-b^2*c^2;
xp=[ x(2)
     (-b*f*s*x(4)^2-e*f*sin(x(1))-b^2*s*c*x(2)^2+b*k*sin(x(3))*c)/im
     x(4)
     (b^2*c*s*x(4)^2+e*b*sin(x(1))*c+a*b*s*x(2)^2-a*k*sin(x(3)))/im ];
```

# Animacija kretanja

```
% koordinate krajeva stapova
x1=l1*sin(fi(:,1));
y1=-l1*cos(fi(:,1));
x2=x1+l2*sin(fi(:,2));
y2=y1-l2*cos(fi(:,2));

fig=figure;
axis([-1.2 1.2 -2 0])
aviobj = avifile('example2.avi')
for i=1:length(t)
  plot([0;x1(i);x2(i)],...
       [0;y1(i);y2(i)],...
       '.-','linewidth',2)
  axis([-1.2 1.2 -2 0])
  F = getframe(fig);
  aviobj = addframe(aviobj,F);
  disp(t(i))
end
aviobj = close(aviobj);

hold on
plot(x1,y1,'r')
plot(x2,y2,'r')
hold off
```