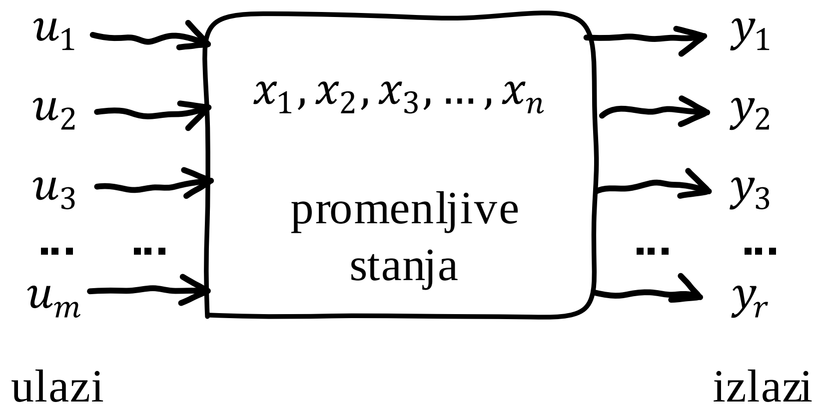
UNIVERZITET U NOVOM SADU
FAKULTET TEHNIČKIH NAUKA
KATEDRA ZA AUTOMATIKU I UPRAVLJANJE SISTEMIMA

Matematički modeli

Modeliranje i simulacija sistema

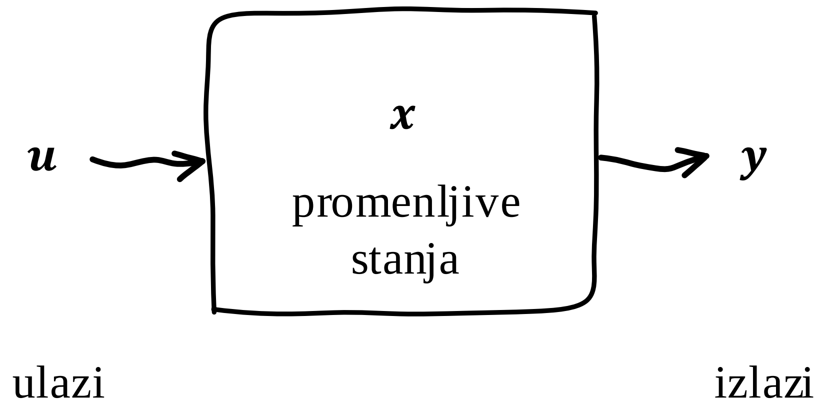
Upravljanje, modelovanje i simulacija sistema

Promenljive za opis modela



- jedan ulaz – jedan izlaz
- više ulaza – *multivarijabilni* model

Upotreba vektora



Tipovi modela

- **Vremenski kontinualni modeli**
  - opisan algebarskim jednačinama
  - opisan običnim diferencijalnim jednačinama

- **Linearni modeli**
- **Vremenski diskretni modeli**
  - Linearan vremenski diskretan model

Model opisan algebarskim jednačinama

- Jednačine koje opisuju vezu ulaza i promenljivih stanja

$$f_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t) = 0$$
$$f_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t) = 0$$
$$\dots$$
$$f_n(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t) = 0$$

Ove jednačine se **rešavaju**.

- Jednačine koje **računaju** izlaze na osnovu ulaza i (prethodno) izračunatih promenljivih stanja

$$y_1(t) = g_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t)$$
$$y_2(t) = g_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t)$$
$$\dots$$
$$y_r(t) = g_r(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t).$$

Model opisan običnim diferencijalnim jednačinama

- Sistem diferencijalnih jednačina prvog reda

$$\frac{dx_1}{dt}(t) = f_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t), \qquad x_1(t_0) = x_{10}$$
$$\frac{dx_2}{dt}(t) = f_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t), \qquad x_2(t_0) = x_{20}.$$
$$\dots$$
$$\frac{dx_n}{dt}(t) = f_n(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t), \qquad x_n(t_0) = x_{n0}$$

Lajbnicova notacija

Ili, drugačije zapisano

$$\dot{x}_1(t) = f_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t), \qquad x_1(t_0) = x_{10}$$
$$\dot{x}_2(t) = f_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t), \qquad x_2(t_0) = x_{20}$$
$$\dots$$
$$\dot{x}_n(t) = f_n(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t), \qquad x_n(t_0) = x_{n0}.$$

Lagranžova notacija

Diferencijalna jednačina višeg reda

- Eksplicitna forma

$$y^{(n)} = f\left(t, y, y', y'', y''', \dots, y^{(n-1)}\right)$$

- Implicitna forma

$$f\left(t, y, y', y'', y''', \dots, y^{(n-1)}, y^{(n)}\right) = 0$$

- Može se pisati sistem jednačina.

- Primer: Van der Pol-ova jednačina

$$y'' - \mu(1 - y^2)y' + y = 0$$

Primer: Lotka-Volter model

Opisuje se ponašanje biološkog sistema: odnos grabljivica-plen. Promenljive: x broj jedinki plena (npr. zečevi), i y kao broj jedinki grabljivica (npr. lisice). Izvodima ovih promenljivih po vremenu su predstavljeni porasti broja populacija, gde parametri $\alpha, \beta, \gamma, \delta$ određuju njihovu međusobnu interakciju.

U formiranju modela su uvedena sledeća pojednostavljenja: da plen uvek ima dovoljno hrane, da zalihe hrane za grabljivice zavise od populacije plena, da je stopa promene populacije proporcionalna njenoj veličini i da se okruženje ne menja u korist jedne vrste i da nema genetske promene.

$$\dot{x} = x(\alpha - \beta y) = \alpha x - \beta x y$$
$$\dot{y} = -y(\gamma - \delta x) = -\gamma y + \delta x y$$

Priraštaj populacije plena raste srazmerno broju jedinki zbog reprodukcije (αx), a opada srazmerno učestanosti susreta sa grabljivicama ($-\beta x y$), gde je ishod fatalan po plen. Promena populacije grabljivica raste srazmerno ishrani ulovljenim plenom ($\delta x y$), a opada zbog prirodne smrti usled nedostatka hrane, koja je srazmerna broju jedinki ($-\gamma y$).

Transformacija diferencijalne jednačine višeg reda u sistem dif. jednačina 1. reda

- Mnogo načina …
- Jedan način: Uvode se smene

$$y_i = y^{(i-1)}, \qquad i = 1,2,\dots,n.$$

$$\begin{aligned} \dot{y} &= y_1 \\ \ddot{y} &= y_2 \\ \dddot{y} &= y_3 \\ &\dots \\ y^{(n-1)} &= y_n \\ y^{(n)} &= f(t, y_1, y_2, \dots, y_n) \end{aligned}$$

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= y_4 \\ &\dots \\ \dot{y}_{n-1} &= y_n \\ \dot{y}_n &= f(t, y_1, y_2, \dots, y_n) \end{aligned}$$

- Primer… (Van der Pol-ova jednačina)

Vektorski zapis

- *Koncept prostora stanja*

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

$$\boldsymbol{y}(t) = \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t), t)$$

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_m \end{bmatrix}, \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_r \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}, \dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dots \\ \dot{x}_n \end{bmatrix}, \boldsymbol{x}(0) = \begin{bmatrix} x_{10} \\ x_{20} \\ \dots \\ x_{n0} \end{bmatrix}$$

Razlozi:

- određivanje rešenja sistema diferencijalnih jednačina prvog reda je brže nego rešavanje odgovarajuće diferencijalne jednačine višeg reda;
- matematičko opisivanje upotrebom vektorske notacije je prilagođeno upotrebi nizova i biblioteka matrične algebre;
- uključivanje početnih uslova je jednostavno;
- ovakav pristup se može primeniti na vremenski promenljive, nelinearne, stohastičke i diskretne modele.

Linearan model

- Linearna algebarska jednačina

$$\sum_{i=1}^{n} a_i(t)x_i + r(t) = 0 \qquad r(t) = b_1u_1(t) + b_2u_2(t) + \cdots + b_mu_m(t)$$

- Linearna diferencijalna jednačina

$$y^{(n)} = \sum_{i=0}^{n-1} a_i(t)y^{(i)} + r(t)$$

ili u razvijenom obliku

$$y^{(n)} = a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0y + r(t)$$

Primer

- Mehanički sistem opisuju dve diferencijalne jednačine 2. reda

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1 x_1 + k(x_1 - x_2) = f(t)$$
$$m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2 x_2 + k(x_2 - x_1) = 0.$$

- ili ekvivalentan sistem od 4 diferencijalne jednačine 1. reda

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -\frac{k_1 + k}{m_1}y_1 - \frac{c_1}{m_1}y_2 + \frac{k}{m_1}y_3 + \frac{1}{m_1}f(t)$$

$$\dot{y}_3 = y_4$$

$$\dot{y}_4 = \frac{k}{m_2}y_1 - \frac{k + k_2}{m_2}y_3 - \frac{c_2}{m_2}y_4$$

Osobine linearnih modela

Opisuju se principima:

1. superpozicije

$$y_1 + y_2 = f(u_1 + u_2) = f(u_1) + f(u_2)$$

$$y_1 = f(u_1), \qquad y_2 = f(u_2)$$

2. homogenosti

$$m \cdot y = f(m \cdot u) = m \cdot f(u)$$

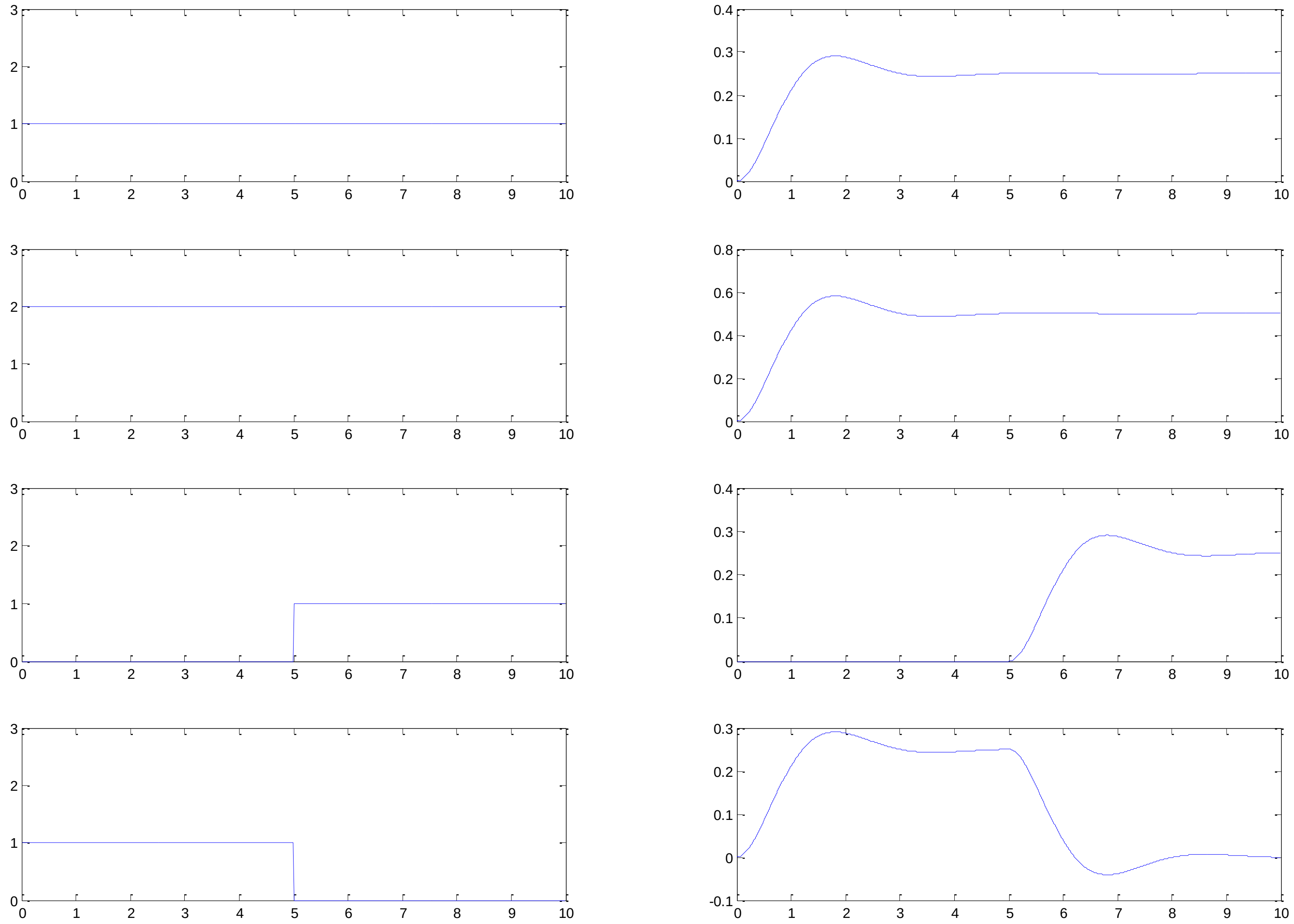
$$y = f(u)$$

3. stacionarnosti

$$y(t - \tau) = f\big(u(t - \tau)\big)$$

$$y = f(u)$$

Linearan model: primeri odziva (desno) na pobudu (levo)



Primer 1

- Da li je model linearan ? $y = au_1 + bu_2$

- Superpozicija: $u_1 = u_{11} + u_{12}, \qquad u_2 = u_{21} + u_{22}$

$$y = a(u_{11}+u_{12}) + b(u_{21}+u_{22})$$

$$y = au_{11} + au_{12} + bu_{21} + bu_{22} = y_{11} + y_{12} + y_{21} + y_{22}$$

$$y = y_1 + y_2$$

- Homogenost: $a(mu_1) + b(mu_2) = m(au_1 + bu_2) = my$

- Stacionarnost: $au_1(t-\tau) + bu_2(t-\tau) = y_1(t-\tau) + y_2(t-\tau) = y(t-\tau)$

- Jeste linearan.

Primer 2

- Da li je linearan? $y = au + b$

- Nije! Ne važi princip homogenosti.

$$m(y) = m(au + b) = m(au) + m(b) \neq a(mu) + b$$

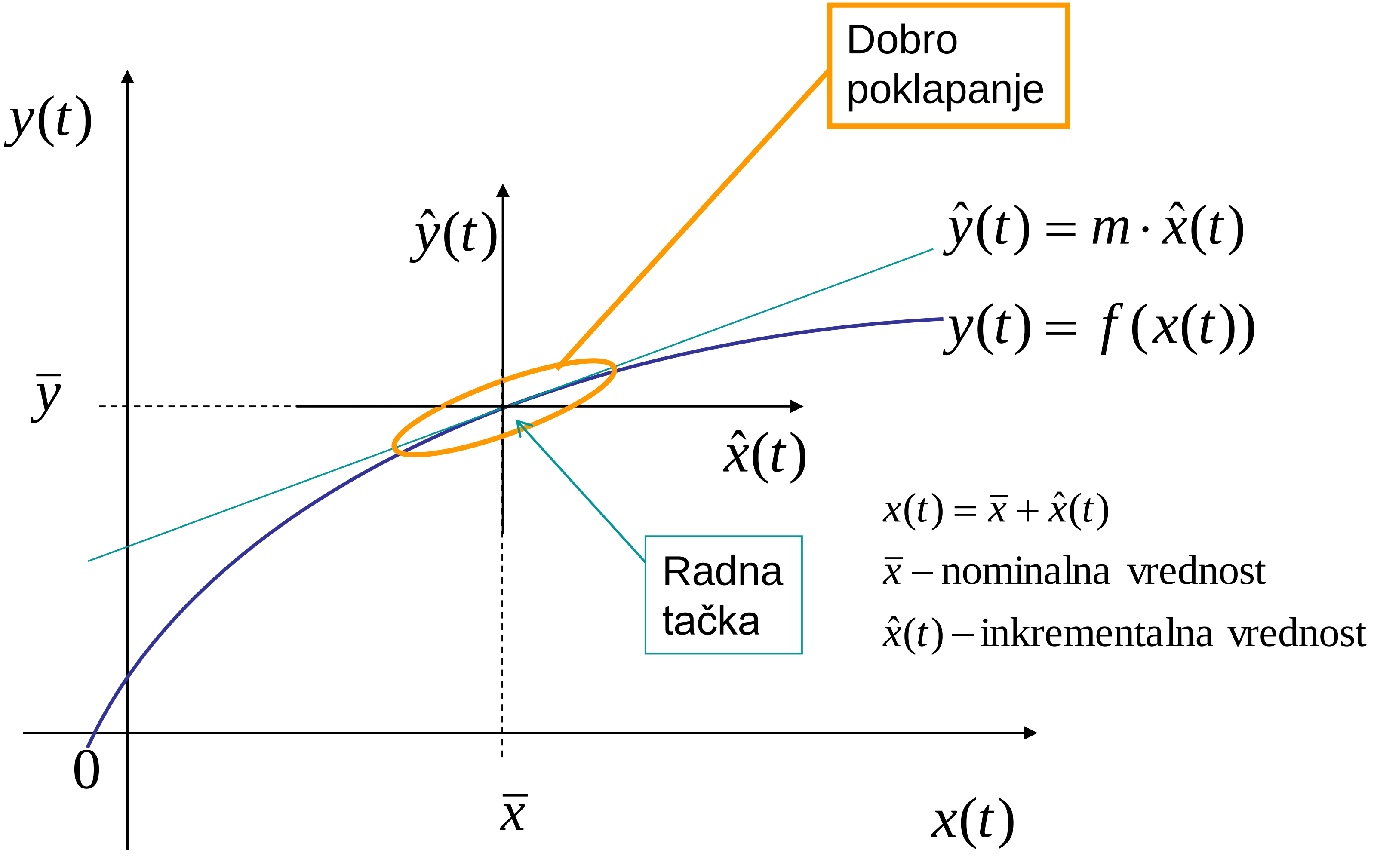
Primer 3

- Da li je linearan? $y = au^2$

- Nije! Ne važi princip superpozicije.

$$a(u_1 + u_2)^2 = au_1^2 + au_2^2 + 2au_1u_2 \neq au_1^2 + au_2^2$$

Princip linearizacije – grafički



Radna tačka

- Radnu tačku u modelu čine vrednosti ulaza, promenljivih stanja i izlaza
- Izbor radne tačke?
  - Najčešće (uobičajene) vrednosti promenljivih
    - Vrednosti u ustaljenom stanju – nominalne vrednosti

- Predstava vrednosti promenljivih zbirom nominalne i inkrementalne vrednosti

$$x_k(t) = \bar{x}_k + \hat{x}_k(t), k = 1,2,\dots,n$$
$$u_k(t) = \bar{u}_k + \hat{u}_k(t), k = 1,2,\dots,m$$
$$y_k(t) = \bar{y}_k + \hat{y}_k(t), k = 1,2,\dots,r$$

Primer linearizacije

- Nelinearan model $y(t) = a \cdot u(t) + b$

- u radnoj tački je $y(t) = \bar{y}$

$$u(t) = \bar{u}$$

$$\bar{y} = a \cdot \bar{u} + b$$

- Uvođenje inkrementalnih vrednosti – smene:

$$y(t) = \bar{y} + \hat{y}(t)$$

$$u(t) = \bar{u} + \hat{u}(t)$$

$$\bar{y} + \hat{y}(t) = a\left(\bar{u} + \hat{u}(t)\right) + b = a\bar{u} + a\hat{u}(t) + b$$

$$\hat{y}(t) = a \cdot \hat{u}(t)$$

← Linearan model

Princip linearizacije – analitički

Razvoj u Tejlorov red funkcije jedne promenljive $y(t) = f(x)$
u okolini radne tačke (nominalne vrednosti)

$$y(t) = f(\bar{x}) + \left.\frac{df}{dx}\right|_{\bar{x}} \frac{x(t) - \bar{x}}{1!} + \left.\frac{d^2 f}{dx^2}\right|_{\bar{x}} \frac{(x(t) - \bar{x})^2}{2!} + \left.\frac{d^3 f}{dx^3}\right|_{\bar{x}} \frac{(x(t) - \bar{x})^3}{3!} + \cdots$$

$$y(t) \approx f(\bar{x}) + \left.\frac{df}{dx}\right|_{\bar{x}} \frac{(x(t) - \bar{x})}{1!}$$

$$y(t) = \bar{y} + \hat{y}(t)$$

$$x(t) = \bar{x} + \hat{x}(t)$$

$$\hat{y}(t) = a\hat{x}(t) \qquad a = \left.\frac{df}{dx}\right|_{\bar{x}}$$

Princip linearizacije – analitički (2)

Razvoj u Tejlorov red funkcije dve promenljive $y(t) = f(\boldsymbol{x})$

$$y(t) = f(\bar{\boldsymbol{x}}) + \left.\frac{\partial f}{\partial x_1}\right|_{\bar{\boldsymbol{x}}} (x_1 - \bar{x}_1) + \left.\frac{\partial f}{\partial x_2}\right|_{\bar{\boldsymbol{x}}} (x_2 - \bar{x}_2)$$

$$+ \frac{1}{2}\left(\left.\frac{\partial^2 f}{\partial x_1^2}\right|_{\bar{\boldsymbol{x}}} (x_1 - \bar{x}_1)^2 + 2 \left.\frac{\partial^2 f}{\partial x_1 \partial x_2}\right|_{\bar{\boldsymbol{x}}} (x_1 - \bar{x}_1)(x_2 - \bar{x}_2) + \left.\frac{\partial^2 f}{\partial x_2^2}\right|_{\bar{\boldsymbol{x}}} (x_2 - \bar{x}_2)^2 \right) + \ldots$$

$$y(t) \approx f(\bar{\boldsymbol{x}}) + \left.\frac{\partial f}{\partial x_1}\right|_{\bar{\boldsymbol{x}}} (x_1 - \bar{x}_1) + \left.\frac{\partial f}{\partial x_2}\right|_{\bar{\boldsymbol{x}}} (x_2 - \bar{x}_2)$$

$$\hat{y}(t) = a_1 \hat{x}_1(t) + a_2 \hat{x}_2(t) \qquad a_1 = \left.\frac{df}{dx_1}\right|_{\bar{\boldsymbol{x}}}, \qquad a_2 = \left.\frac{df}{dx_2}\right|_{\bar{\boldsymbol{x}}}.$$

Princip linearizacije – analitički (3)

Razvoj u Tejlorov red funkcije više promenljivih $y(t) = f(\boldsymbol{x})$

$$y(t) \approx f(\overline{\boldsymbol{x}}) + \left.\frac{\partial f}{\partial x_1}\right|_{\overline{\boldsymbol{x}}} (x_1 - \bar{x}_1) + \left.\frac{\partial f}{\partial x_2}\right|_{\overline{\boldsymbol{x}}} (x_2 - \bar{x}_2) + \cdots + \left.\frac{\partial f}{\partial x_n}\right|_{\overline{\boldsymbol{x}}} (x_n - \bar{x}_n)$$

$$\hat{y}(t) = a_1 \hat{x}_1(t) + a_2 \hat{x}_2(t) + \cdots + a_n \hat{x}_n(t)$$

Model u prostoru stanja (još jednom)

- Pomena (kretanje) promenljivih stanja

$$\dot{x}_1(t) = f_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t), \qquad x_1(t_0) = x_{10}$$
$$\dot{x}_2(t) = f_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t), \qquad x_2(t_0) = x_{20}$$
$$\dots$$
$$\dot{x}_n(t) = f_n(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t), \qquad x_n(t_0) = x_{n0}$$

- Izlazi modela

$$y_1(t) = g_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t)$$
$$y_2(t) = g_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t)$$
$$\dots$$
$$y_r(t) = g_r(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t)$$

Linearizacija (nelinearnog) modela u prostoru stanja

Izvod promenljive stanja: $\dot{x}_i(t) = f_i(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t)$

$$\dot{x}_i \approx f_i\Big|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}} + \frac{\partial f_i}{\partial x_1}\bigg|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}} (x_1 - \bar{x}_1) + \cdots + \frac{\partial f_i}{\partial x_n}\bigg|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}} (x_n - \bar{x}_n)$$
$$+ \frac{\partial f_i}{\partial u_1}\bigg|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}} (u_1 - \bar{u}_1) + \cdots + \frac{\partial f_i}{\partial u_m}\bigg|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}} (u_m - \bar{u}_m)$$

$$\dot{\hat{x}}_i = \dot{x}_i - f_i\Big|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}} = \frac{\partial f_i}{\partial x_1}\bigg|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}} \hat{x}_1 + \cdots + \frac{\partial f_i}{\partial x_n}\bigg|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}} \hat{x}_n + \frac{\partial f_i}{\partial u_1}\bigg|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}} \hat{u}_1 + \cdots + \frac{\partial f_i}{\partial u_m}\bigg|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}} \hat{u}_m$$

$$\dot{\hat{x}}_i(t) = a_{i1}\hat{x}_1(t) + \cdots + a_{in}\hat{x}_n(t) + b_{i1}\hat{u}_1(t) + \cdots + b_{im}\hat{u}_m(t)$$

Sistem običnih linearnih diferencijalnih jednačina 1. reda

$$\dot{\hat{x}}_1(t) = a_{11}\hat{x}_1(t) + \cdots + a_{1n}\hat{x}_n(t) + b_{11}\hat{u}_1(t) + \cdots + b_{1m}\hat{u}_m(t)$$
$$\dot{\hat{x}}_2(t) = a_{21}\hat{x}_1(t) + \cdots + a_{2n}\hat{x}_n(t) + b_{21}\hat{u}_1(t) + \cdots + b_{2m}\hat{u}_m(t)$$
$$\dots$$
$$\dot{\hat{x}}_n(t) = a_{n1}\hat{x}_1(t) + \cdots + a_{nn}\hat{x}_n(t) + b_{n1}\hat{u}_1(t) + \cdots + b_{nm}\hat{u}_m(t)$$

$$\hat{x}_1(0) = \hat{x}_{10}$$
$$\hat{x}_2(0) = \hat{x}_{20}$$
$$\dots$$
$$\hat{x}_n(0) = \hat{x}_{n0}$$

$$a_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\substack{\bar{\boldsymbol{x}} \\ \bar{\boldsymbol{u}}}}, \qquad b_{ik} = \left.\frac{\partial f_i}{\partial u_k}\right|_{\substack{\bar{\boldsymbol{x}} \\ \bar{\boldsymbol{u}}}}, \qquad i = 1,2,\dots,n, \qquad j = 1,2,\dots,n, \qquad k = 1,2,\dots,m.$$

Linearizacija (nelinearnog) modela u prostoru stanja (2)

Izlaz: $y_i(t) = g_i(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_m; t)$

$$y_i \approx g_i\Big|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}} + \frac{\partial g_i}{\partial x_1}\Big|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}}(x_1 - \bar{x}_1) + \cdots + \frac{\partial g_i}{\partial x_n}\Big|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}}(x_n - \bar{x}_n)$$
$$+ \frac{\partial g_i}{\partial u_1}\Big|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}}(u_1 - \bar{u}_1) + \cdots + \frac{\partial g_i}{\partial u_m}\Big|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}}(u_m - \bar{u}_m)$$

$$\hat{y}_i = y_i - g_i\Big|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}} = \frac{\partial g_i}{\partial x_1}\Big|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}}\hat{x}_1 + \cdots + \frac{\partial g_i}{\partial x_n}\Big|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}}\hat{x}_n + \frac{\partial g_i}{\partial u_1}\Big|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}}\hat{u}_1 + \cdots + \frac{\partial g_i}{\partial u_m}\Big|_{\substack{\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}}}\hat{u}_m$$

$$\hat{y}_i(t) = c_{i1}\hat{x}_1(t) + \cdots + c_{in}\hat{x}_n(t) + d_{i1}\hat{u}_1(t) + \cdots + d_{im}\hat{u}_m(t)$$

Sistem linearnih algebarskih jednačina

$$
\begin{gathered}
\hat{y}_1(t) = c_{11}\hat{x}_1(t) + \cdots + c_{1n}\hat{x}_n(t) + d_{11}\hat{u}_1(t) + \cdots + d_{1m}\hat{u}_m(t) \\
\hat{y}_2(t) = c_{21}\hat{x}_1(t) + \cdots + c_{2n}\hat{x}_n(t) + d_{21}\hat{u}_1(t) + \cdots + d_{2m}\hat{u}_m(t) \\
\dots \\
\hat{y}_r(t) = c_{r1}\hat{x}_1(t) + \cdots + c_{rn}\hat{x}_n(t) + d_{r1}\hat{u}_1(t) + \cdots + d_{rm}\hat{u}_m(t)
\end{gathered}
$$

$$
c_{ij} = \left.\frac{\partial g_i}{\partial x_j}\right|_{\substack{\bar{\boldsymbol{x}} \\ \bar{\boldsymbol{u}}}}, \qquad d_{ik} = \left.\frac{\partial g_i}{\partial u_k}\right|_{\substack{\bar{\boldsymbol{x}} \\ \bar{\boldsymbol{u}}}}, \qquad i = 1,2,\dots,r, \qquad j = 1,2,\dots,n, \qquad k = 1,2,\dots,m.
$$

Linearan matematički model u prostoru stanja

(u razvijenom obliku)

$$\begin{aligned}
\dot{\hat{x}}_1(t) &= a_{11}\hat{x}_1(t) + \cdots + a_{1n}\hat{x}_n(t) + b_{11}\hat{u}_1(t) + \cdots + b_{1m}\hat{u}_m(t) \\
\dot{\hat{x}}_2(t) &= a_{21}\hat{x}_1(t) + \cdots + a_{2n}\hat{x}_n(t) + b_{21}\hat{u}_1(t) + \cdots + b_{2m}\hat{u}_m(t) \\
&\dots \\
\dot{\hat{x}}_n(t) &= a_{n1}\hat{x}_1(t) + \cdots + a_{nn}\hat{x}_n(t) + b_{n1}\hat{u}_1(t) + \cdots + b_{nm}\hat{u}_m(t)
\end{aligned}$$

$$\begin{aligned}
\hat{x}_1(0) &= \hat{x}_{10} \\
\hat{x}_2(0) &= \hat{x}_{20} \\
&\dots \\
\hat{x}_n(0) &= \hat{x}_{n0}
\end{aligned}$$

$$\begin{aligned}
\hat{y}_1(t) &= c_{11}\hat{x}_1(t) + \cdots + c_{1n}\hat{x}_n(t) + d_{11}\hat{u}_1(t) + \cdots + d_{1m}\hat{u}_m(t) \\
\hat{y}_2(t) &= c_{21}\hat{x}_1(t) + \cdots + c_{2n}\hat{x}_n(t) + d_{21}\hat{u}_1(t) + \cdots + d_{2m}\hat{u}_m(t) \\
&\dots \\
\hat{y}_r(t) &= c_{r1}\hat{x}_1(t) + \cdots + c_{rn}\hat{x}_n(t) + d_{r1}\hat{u}_1(t) + \cdots + d_{rm}\hat{u}_m(t)
\end{aligned}$$

Linearan matematički model u prostoru stanja

(u vektorskom obliku)

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{A} \cdot \hat{\boldsymbol{x}}(t) + \boldsymbol{B} \cdot \hat{\boldsymbol{u}}(t), \qquad \hat{\boldsymbol{x}}(0) = \boldsymbol{x}_0$$

$$\hat{\boldsymbol{y}}(t) = \boldsymbol{C} \cdot \hat{\boldsymbol{x}}(t) + \boldsymbol{D} \cdot \hat{\boldsymbol{u}}(t)$$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \dots & \dots & & \dots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & & \dots \\ c_{r1} & c_{r2} & \dots & c_{rn} \end{bmatrix} \quad \boldsymbol{D} = \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1m} \\ d_{21} & d_{22} & \dots & d_{2m} \\ \dots & \dots & & \dots \\ d_{r1} & d_{r2} & \dots & d_{rm} \end{bmatrix}$$

$$\hat{\boldsymbol{u}} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \dots \\ \hat{u}_m \end{bmatrix}, \qquad \hat{\boldsymbol{y}} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \dots \\ \hat{y}_r \end{bmatrix}, \qquad \hat{\boldsymbol{x}} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \dots \\ \hat{x}_n \end{bmatrix}.$$

Pisanje “kapica” se izbegava

(In)varijantan model

- Invarijantan model

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{A} \cdot \hat{\boldsymbol{x}}(t) + \boldsymbol{B} \cdot \hat{\boldsymbol{u}}(t), \qquad \hat{\boldsymbol{x}}(0) = \boldsymbol{x}_0$$

$$\hat{\boldsymbol{y}}(t) = \boldsymbol{C} \cdot \hat{\boldsymbol{x}}(t) + \boldsymbol{D} \cdot \hat{\boldsymbol{u}}(t)$$

- Vremenski promenljiv model: $\boldsymbol{A} \equiv \boldsymbol{A}(t), \boldsymbol{B} \equiv \boldsymbol{B}(t), \boldsymbol{C} \equiv \boldsymbol{C}(t), \boldsymbol{D} \equiv \boldsymbol{D}(t)$

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{A}(t) \cdot \hat{\boldsymbol{x}}(t) + \boldsymbol{B}(t) \cdot \hat{\boldsymbol{u}}(t), \qquad \hat{\boldsymbol{x}}(0) = \boldsymbol{x}_0$$

$$\hat{\boldsymbol{y}}(t) = \boldsymbol{C}(t) \cdot \hat{\boldsymbol{x}}(t) + \boldsymbol{D}(t) \cdot \hat{\boldsymbol{u}}(t)$$

Određivanje radne tačke

- nominalne vrednosti (promenljivih stanja, ulaza i izlaza) u radnoj tački su nekada unapred poznate
  - mogu se dobiti merenjem
- deo tih vrednosti se može izračunati na osnovu poznavanja ostalih nominalnih vrednosti

$$
\begin{gathered}
0 = f_1(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n; \bar{u}_1, \bar{u}_2, \dots, \bar{u}_m) \\
0 = f_2(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n; \bar{u}_1, \bar{u}_2, \dots, \bar{u}_m) \\
\dots \\
0 = f_n(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n; \bar{u}_1, \bar{u}_2, \dots, \bar{u}_m)
\end{gathered}
$$

$$
\begin{gathered}
\bar{y}_1 = g_1(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n; \bar{u}_1, \bar{u}_2, \dots, \bar{u}_m) \\
\bar{y}_2 = g_2(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n; \bar{u}_1, \bar{u}_2, \dots, \bar{u}_m) \\
\dots \\
\bar{y}_r = g_r(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n; \bar{u}_1, \bar{u}_2, \dots, \bar{u}_m).
\end{gathered}
$$

Koraci linearizacije

1. Odrediti radnu tačku – pisanjem i rešavanjem odgovarajućih algebarskih jednačina
2. Prepisati sve linearne članove kao sume nominalne i inkrementalne vrednosti
3. Zameniti sve nelinearne članove sa prva 2 sabirka razvoja u Tejlovor red
4. Skratiti konstantne članove u differencijalnim jednačinama (Upotrebiti algebarske jednačine koje ordeđuju radnu tačku.)
5. Definisati početne vrednosti inkrementalnih promenljivih

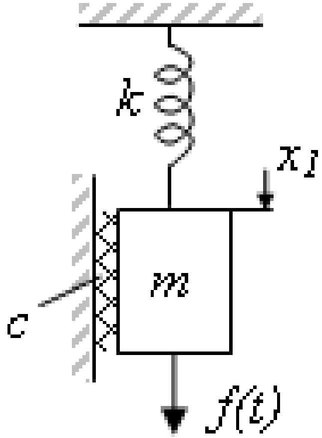
$$\hat{x}(0) = x(0) - \bar{x}$$

Izbor promenljivih stanja linearnog modela

- promenljive stanja moraju da čine minimalan skup linearno nezavisnih promenljivih
- pogodno je da promenljive stanja imaju fizičku interpretaciju
- obično se za promenljive stanja biraju fizičke veličine elemenata sposobnih da prime i uskladište energiju
- kada se ponašanje sistema opisuje diferencijalnom jednačinom višeg reda, dobar kandidat za promenljivu stanja je veličina za koju se računaju izvodi

Primer

- Amortizer se kreće pod dejstvom gravitacione sile i pobudne sile $f(t)$. Međutim, ne može se smatrati da je sila u opruzi linearna za sva istezanja, nego je ona funkcija pomeraja x, $f_k = f_k(x)$. Ukoliko je ta sila $f_k(x) = kx^3$ i pobudna sila $f_0 = const$ odrediti vrednost pomeraja x_0. Nadalje, ako se pobudna sila menja kao $f(t) = f_0 + \Delta f(t)$, napisati linearan matematički model koji opisuje promene položaja u odnosu na nominalnu vrednost x_0.



$$m\ddot{x} + c\dot{x} + f_k(x) = mg + f(t)$$

Primer (2)

- U ustaljenom stanju, promene položaja x ne postoje i tada važi

$$kx_0^3 = mg + f_0 \qquad x_0 = \sqrt[3]{\frac{mg + f_0}{k}}$$

- radna tačka

$$f(t) = f_0 + \Delta f(t) = \bar{f} + \hat{f}(t)$$

$$x(t) = x_0 + \Delta x(t) = \bar{x} + \hat{x}(t)$$

$$m\ddot{\hat{x}} + c\dot{\hat{x}} + k(\bar{x} + \hat{x})^3 = mg + \bar{f} + \hat{f}(t)$$

- Linearizacija nelinearnog člana $F_k(x)=kx^3$

$$F_k(\bar{x} + \hat{x}(t)) \approx F(\bar{x}) + \left.\frac{\partial F(x)}{\partial x}\right|_{x=\bar{x}} \hat{x}(t) = k\bar{x}^3 + 3k\bar{x}^2\hat{x}(t)$$

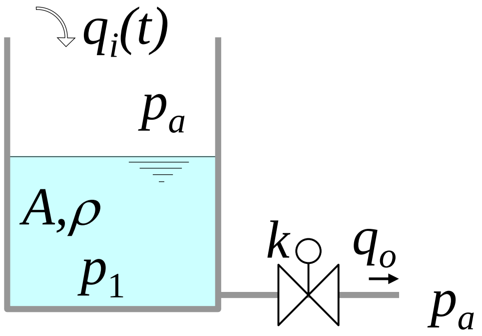
- Nakon zamene

$$m\ddot{\hat{x}} + c\dot{\hat{x}} + k\bar{x}^3 + 3k\bar{x}^2\hat{x} = mg + \bar{f} + \hat{f}$$

$$m\ddot{\hat{x}} + c\dot{\hat{x}} + k_1\hat{x} = \hat{f} \qquad k_1 = 3k\bar{x}^2$$

Primer: Dinamički model hidrauličkog sistema

- Primer: Napisati nelinearan model promene p_1 i linearizovati ga



$$\dot{p}_1 = \frac{1}{C}\left[q_{in}(t) - q_{out}(t)\right]$$

$$q_{in}(t) = q_i(t)$$

$$q_{out}(t) = k\sqrt{p_1 - p_a}$$

$$\dot{p}_1 = \frac{1}{C}\left[-k\sqrt{p_1 - p_a} + q_i(t)\right]$$

Linearizacija modela

1. U ustaljenom stanju: $\dot{p}_1 = 0 \quad \Rightarrow \quad k\sqrt{\overline{p}_1 - p_a} = \overline{q}_i \qquad \overline{p}_1 = p_a + \frac{1}{k^2}\cdot\overline{q}_i^{\,2}$
2. Predstava linearnih članova: $p_1(t) = \overline{p}_1 + \hat{p}_1(t) \quad q_i(t) = \overline{q} + \hat{q}_i(t)$
3. Linearzicija nelinearnog člana i
4. skraćivanje konstantnih članova
   - Na osnovu: $\sqrt{x+\Delta x} \approx \sqrt{x}\Big|_{x_0} + \frac{1}{2\sqrt{x}}\Big|_{x_0} \Delta x + \ldots$
   - Dobijamo:

$$\dot{\hat{p}}_1(t) = \frac{1}{C}\left[-k\sqrt{\hat{p}_1(t) + \overline{p}_1 - p_a} + \overline{q}_i + \hat{q}_i(t)\right] =$$

$$= \frac{1}{C}\left[-k\left(\sqrt{\overline{p}_1 - p_a} + \frac{1}{2\sqrt{\overline{p}_1 - p_a}}\,\hat{p}_1(t)\right) + \overline{q}_i + \hat{q}_i(t)\right] =$$

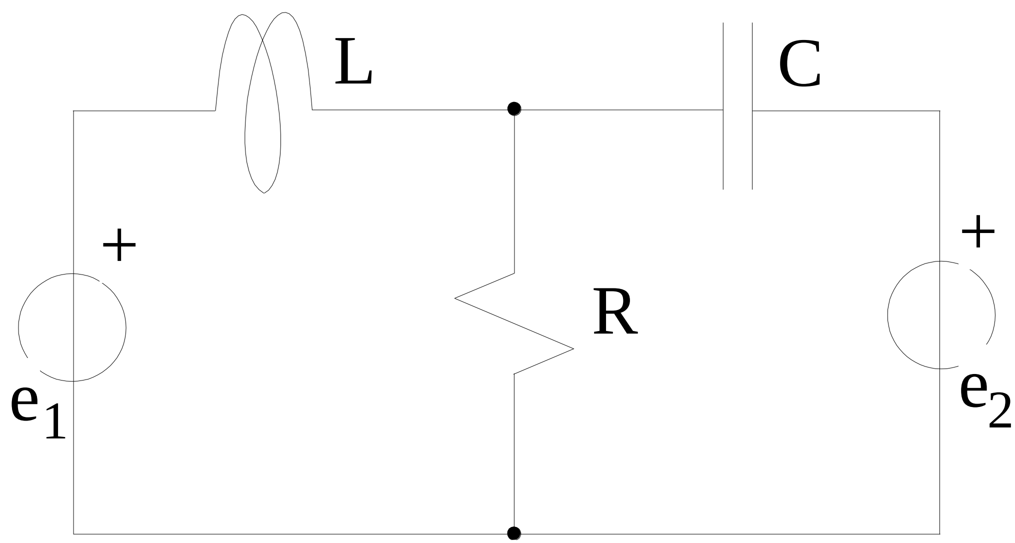
$$= \frac{-k}{2C}\,\frac{k}{\overline{q}_i}\,\hat{p}_1(t) + \frac{1}{C}\,\hat{q}_i(t)$$

   - Konačno:

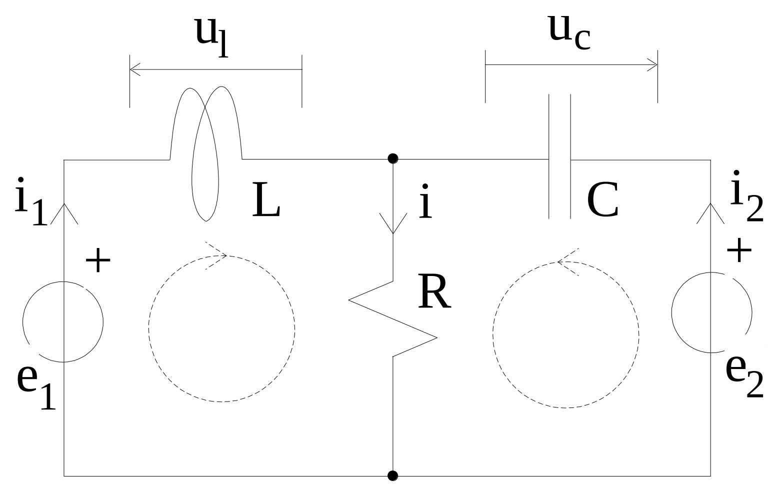
$$\dot{\hat{p}}_1(t) + \frac{1}{RC}\,\hat{p}_1(t) = \frac{1}{C}\,\hat{q}_i(t) \qquad R = \frac{2\overline{q}_i}{k^2}$$

Primer linearnog modela

a) Formirati matematički model u prostoru stanja za električno kolo na slici.
b) Ako se za izlaznu velicinu usvoji struja kroz otpornik R, formirati jednačinu izlaza.
c) Formirati jednačinu izlaza ako se za izlazne veličine usvoje struje kroz izvore e_1 i e_2.



Primer (a)



$$e_1 = u_L + Ri = L\frac{di_1}{dt} + Ri_1 + Ri_2$$

$$e_2 = u_C + Ri = u_C + Ri_1 + Ri_2$$

$$i_2 = C\frac{du_C}{dt} \qquad L\frac{di_1}{dt} + RC\frac{du_c}{dt} = e_1 - Ri_1$$

$$RC\frac{du_C}{dt} = e_2 - u_C - Ri_1$$

$$L\frac{di_1}{dt} = e_1 - Ri_1 - RC\frac{du_C}{dt} = e_1 - Ri_1 - e_2 + u_C + Ri_1 = u_c + e_1 - e_2$$

$$\frac{di_1}{dt} = \frac{1}{L}u_C + \frac{1}{L}e_1 - \frac{1}{L}e_2$$

$$\frac{du_C}{dt} = \frac{1}{RC}e_2 - \frac{1}{RC}u_C - \frac{1}{C}i_1$$

$$\begin{bmatrix} \frac{di_1}{dt} \\ \frac{du_c}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} i_1 \\ u_C \end{bmatrix} + \begin{bmatrix} \frac{1}{L} & -\frac{1}{L} \\ 0 & \frac{1}{RC} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$\mathrm{A} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \qquad \mathrm{B} = \begin{bmatrix} \frac{1}{L} & -\frac{1}{L} \\ 0 & \frac{1}{RC} \end{bmatrix}$$

Primer (b, c)

b) ...

$$i = i_1 + i_2 \; ; \quad \mathrm{i}_2 = C\frac{du_C}{dt} = \frac{1}{R}e_2 - \frac{1}{R}u_C - i_1$$

$$i = i_1 + \frac{1}{R}e_2 - \frac{1}{R}u_C - i_1 = -\frac{1}{R}u_C + \frac{1}{R}e_2$$

$$i = \begin{bmatrix} 0 & -\frac{1}{R} \end{bmatrix} \begin{bmatrix} i_1 \\ u_C \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{R} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

c) ...

$$i_1 = i_1$$

$$i_2 = -i_1 - \frac{1}{R}u_C + \frac{1}{R}e_2$$

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -\frac{1}{R} \end{bmatrix} \begin{bmatrix} i_1 \\ u_C \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{R} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

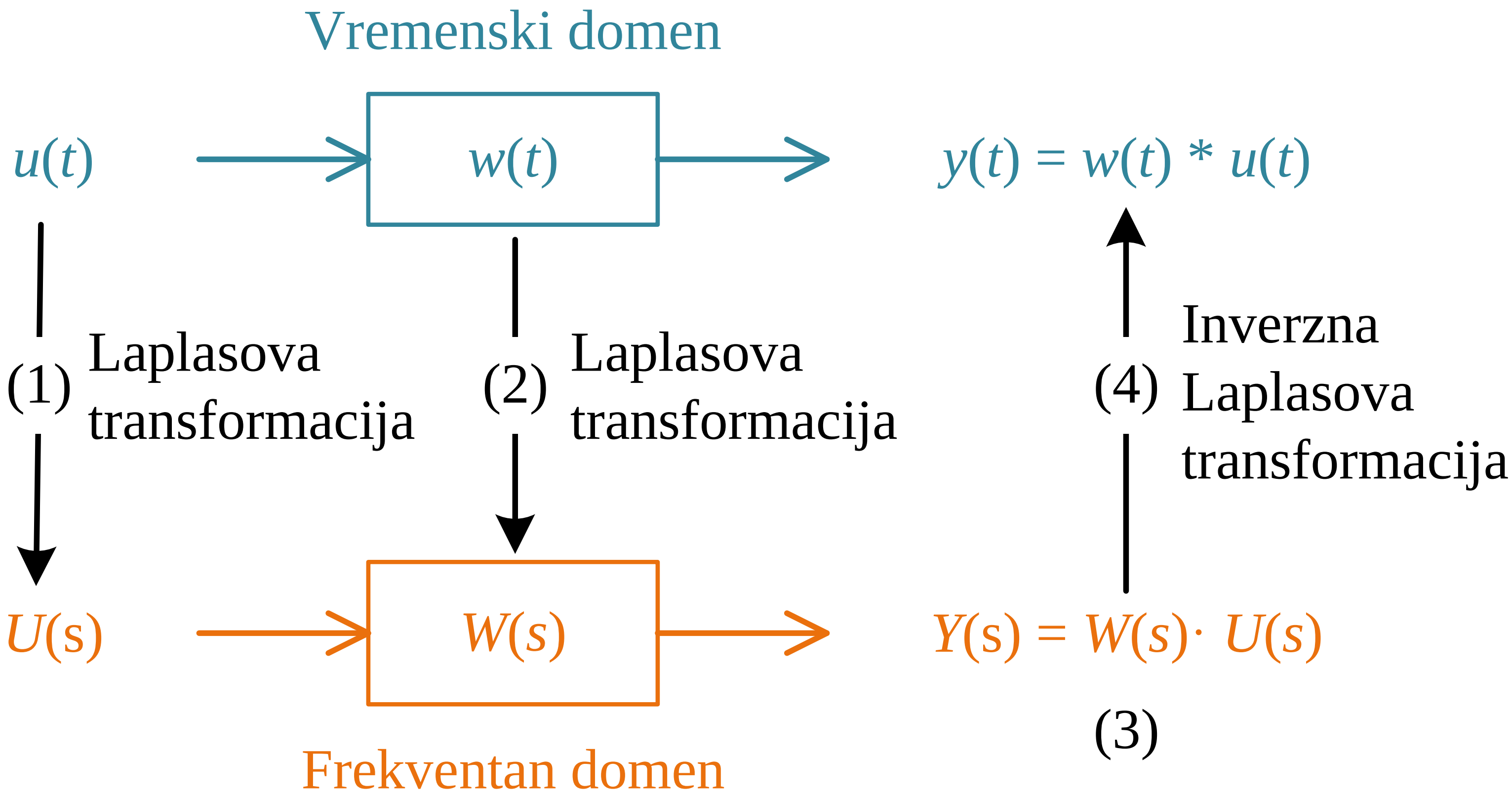
Laplasova transformacija

Pierre-Simon Laplace

Pierre-Simon Laplace (1749–1827). Posthumous portrait by Jean-Baptiste Paulin Guérin, 1838.

Born	(1749-03-23)23 March 1749 Beaumont-en-Auge, Normandy, Kingdom of France
Died	5 March 1827(1827-03-05) (aged 77) Paris, Bourbon France
Nationality	French
Alma mater	University of Caen
Known for	Laplace transform …
Scientific career	
Fields	Astronomer and mathematician
Institutions	École Militaire (1769–1776)
Academic advisors	Jean d'Alembert Christophe Gadbled Pierre Le Canu
Doctoral students	Siméon Denis Poisson

Primena Laplasove transformacije



Definicija Laplasove transformacije

- Definicija

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0}^{\propto} f(t)e^{-st}dt$$

  - $\mathcal{L}$ – operator Laplasove transformacije,
  - s – kompleksna učestanost ili kompleksna promenljiva Laplasove transformacije,
  - $f(t)$ – vremenski zavisan signal,
  - $F(s)$ – kompleksan lik signala $f(t)$.

- Inverzna Laplasova transformacija

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\gamma-j\omega}^{\gamma+j\omega} F(s)e^{st}ds, \qquad t > 0$$

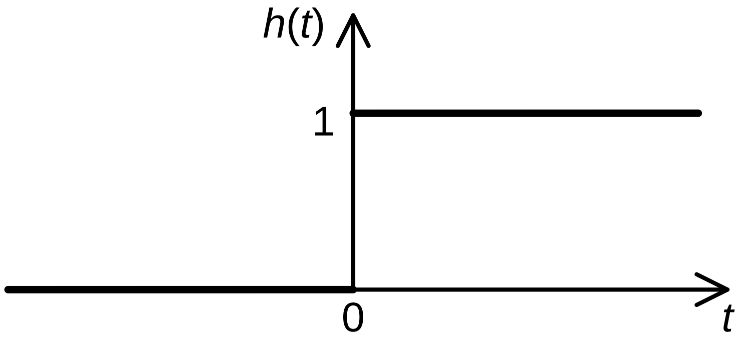
Tablica Laplasove transformacije

Br.	Original $f(t), (t > 0)$	Kompleksan lik $F(s)$
1	$\delta(t)$	1
2	$t^n, \qquad n = 0,1,2,\dots$	$\frac{n!}{s^{n+1}}$
3	e^{-at}	$\frac{1}{s+a}$
4	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
5	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
6	$\mathrm{e}^{-\alpha \mathrm{t}}\cos(\omega t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$
7	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
8	$\mathrm{e}^{-\alpha \mathrm{t}}\sin(\omega t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$

Osobine Laplasove transformacije

Br.	Naziv osobine	Izraz
1	Linearnost	$\mathcal{L}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 F_1(s) + a_2 F_2(s)$
2	Čisto vremensko kašnjenje	$\mathcal{L}\{f(t-\tau)\} = e^{-s\tau} F(s)$
3	Pomeranje kompleksnog lika	$\mathcal{L}\{e^{-at} f(t)\} = F(s+a)$
4	Konvolucija originala	$\mathcal{L}\{f_1(t) * f_2(t)\} = F_1(s) F_2(s)$
5	Izvod originala	$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0)$ $\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$
6	Integral originala	$\mathcal{L}\left\{\int_0^t f(t)dt\right\} = \frac{F(s)}{s}$ $\mathcal{L}\left\{\iint_0^t \dots \int_0^t f(t)dt^n\right\} = \frac{F(s)}{s^n}$
7	Izvod kompleksnog lika	$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$
8	Promena vremenske skale	$\mathcal{L}\left\{f\left(\frac{t}{a}\right)\right\} = aF(as)$
9	Granične vrednosti	$\lim_{t \to 0} f(t) = \lim_{s \to \propto} sF(s)$ $\lim_{t \to \propto} f(t) = \lim_{s \to 0} sF(s)$

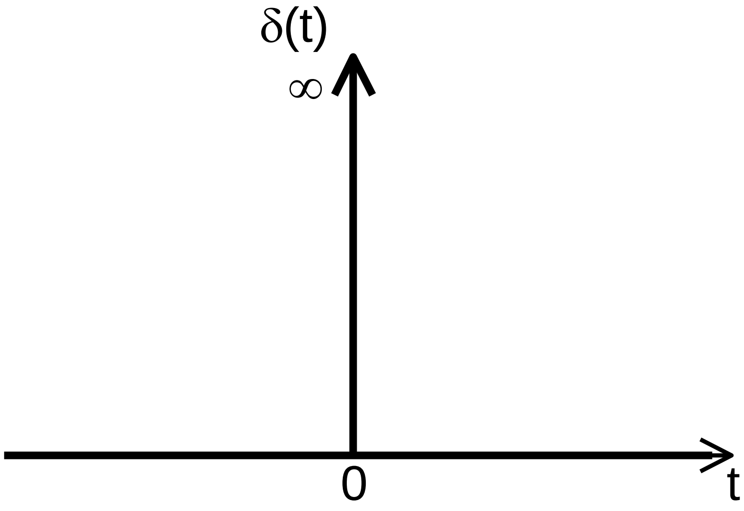
Hevisajdov signal



$$h(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

$$F(s) = L\{f(t)\} = \int_{0-}^{\infty} f(t) e^{-st} dt = \int_{0-}^{\infty} 1 \cdot e^{-st} dt = -\left.\frac{e^{-st}}{s}\right|_{0}^{\infty} = \frac{1}{s}$$

Dirakov impuls

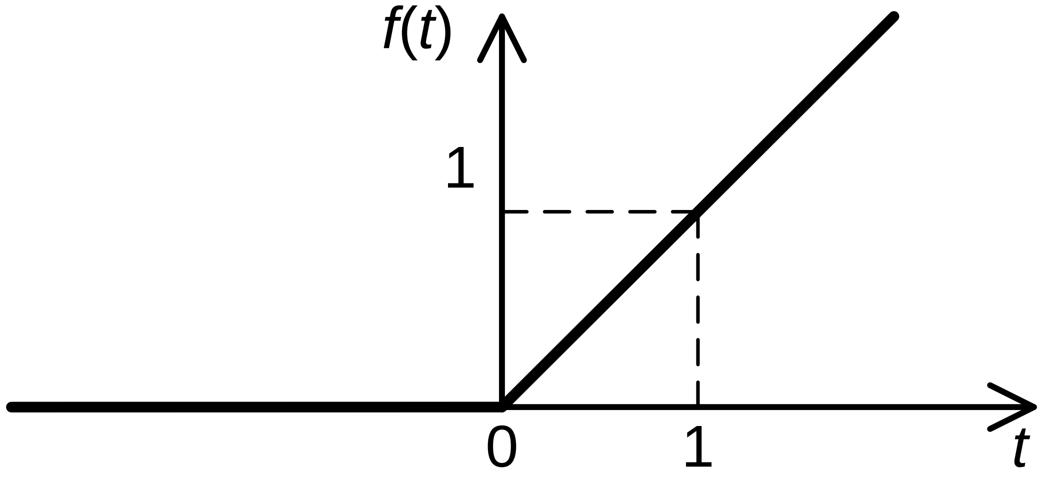


$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$F(s) = L\{\frac{dh(t)}{dt}\} = sL\{h(t)\} - h(0_{-}) = s \cdot \frac{1}{s} - 0 = 1$$

Jedinični nagibni signal

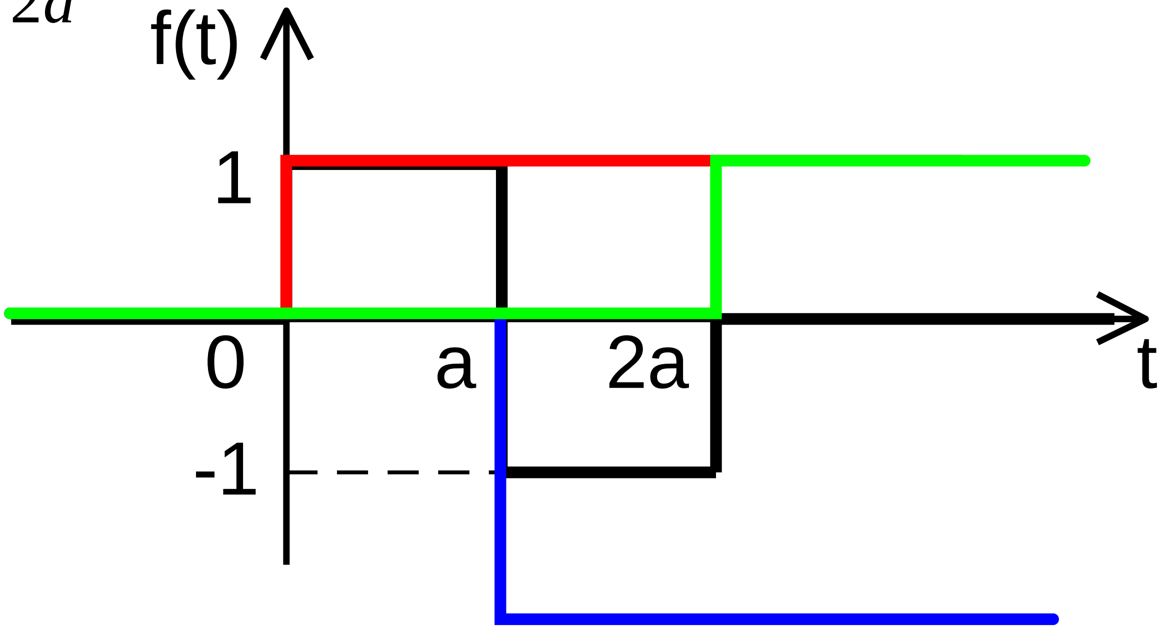


$$h(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$$

$$F(s) = L\{f(t)\} = L\{t \cdot h(t)\} = \int_0^{\infty} t \cdot e^{-st} dt = -\left.\frac{t \cdot e^{-st}}{s}\right|_0^{\infty} + \frac{1}{s}\int_0^{\infty} e^{-st} dt = \frac{1}{s^2}$$

Primer – složen signal 1

$$f(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 \le t < a \\ -1, & a \le t < 2a \\ 0, & t \ge a \end{cases}$$

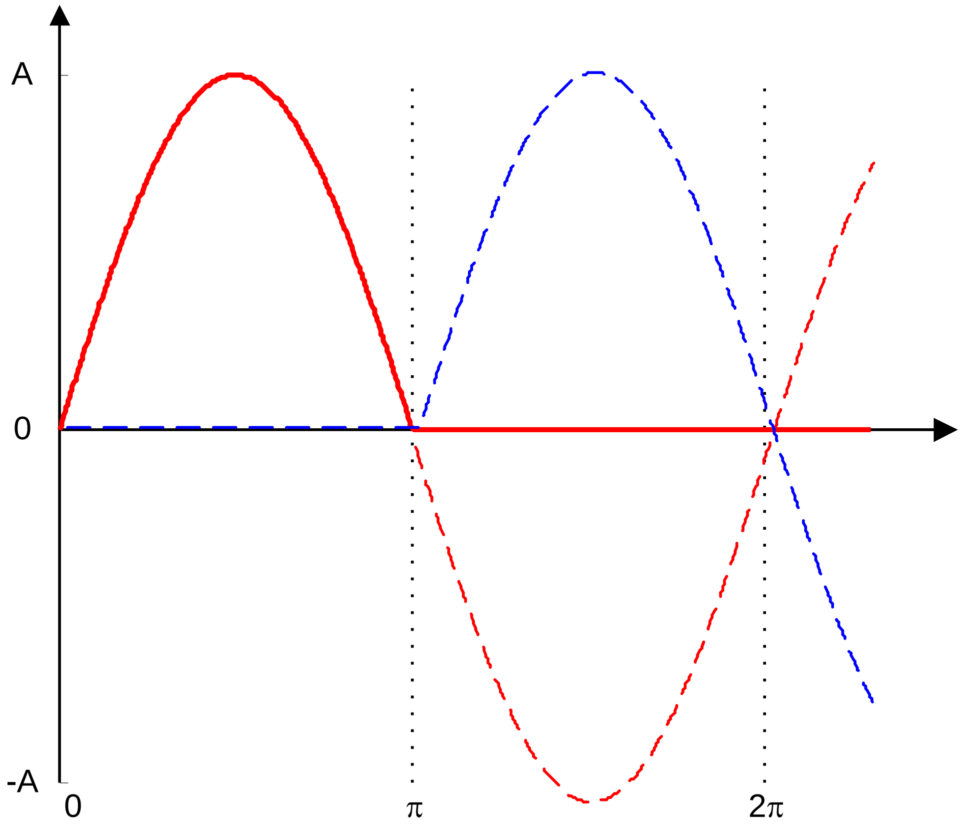


$$f(t) = h(t) - 2h(t-a) + h(t-2a)$$

$$F(s) = L\{f(t)\} = \frac{1}{s}\left(1 - 2e^{-as} + e^{-2as}\right)$$

Primer – složen signal 2

$$f(t) = \begin{cases} 0, & t < 0 \\ A\sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$$

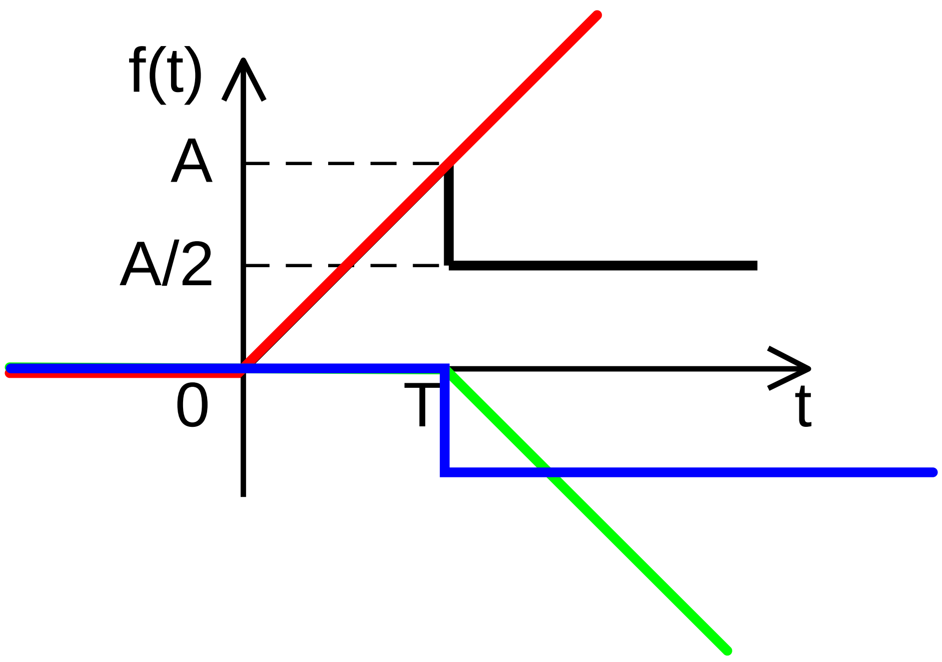


$$f(t) = A\sin t \cdot h(t) + A\sin(t-\pi) \cdot h(t-\pi)$$

$$F(s) = \frac{A}{s^2+1} + \frac{Ae^{-s\pi}}{s^2+1} = \frac{A}{s^2+1}\left(1+e^{-s\pi}\right)$$

Primer – složen signal 3

$$f(t)=\begin{cases}0, & t<0\\ \dfrac{A}{T}t, & 0\le t<T\\ \dfrac{A}{2}, & t\ge T\end{cases}$$



$$f(t)=\frac{A}{T}t\cdot h(t)-\frac{A}{2}h(t-T)-\frac{A}{T}(t-T)\cdot h(t-T)$$

$$F(s)=\frac{A}{Ts^2}\left(1-e^{-sT}\right)-\frac{A}{2s}e^{-sT}$$

Inverzna Laplasova transformacija

$$F(s) = \frac{P(s)}{Q(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

- Za određivanje inverzne Laplasove transformacije su od posebnog značaja polovi funkcije *F*(*s*), i tu se mogu uočiti četiri karakteristična slučaja:
  - Svi polovi funkcije *F*(*s*) su realni i prosti
  - Funkcija *F*(*s*) ima višestruke realne korene
  - Postoje konjugovano-kompleksni polovi, a realni su, ako postoje, prosti
  - Funkcija *F*(*s*) ima višestruke konjugovano kompleksne polove

Polovi funkcije su realni i jednostruki

$$F(s)=\frac{P(s)}{Q(s)}=\frac{P(s)}{(s-s_1)(s-s_2)\ldots(s-s_n)}$$

$$F(s)=\frac{K_1}{s-s_1}+\frac{K_2}{s-s_2}+\ldots+\frac{K_n}{s-s_n}=\sum_{k=1}^{n}\frac{K_k}{s-s_k}$$

$$K_k=\left[(s-s_k)\frac{P(s)}{Q(s)}\right]_{s=s_k}$$

$$K_k=\lim_{s\to s_k}\left[(s-s_k)\frac{P(s)}{Q(s)}\right]=\lim_{s\to s_k}\frac{\frac{d}{ds}(s-s_k)P(s)}{\frac{d}{ds}Q(s)}=\frac{P(s_k)}{Q'(s_k)}$$

$$f(t)=L^{-1}\left[\sum_{k=1}^{n}\frac{K_k}{s-s_k}\right]=\sum_{k=1}^{n}K_k\cdot e^{s_k t},\quad t>0$$

Polovi funkcije su realni višestruki

$$F(s)=\frac{P(s)}{Q(s)}=\frac{P(s)}{(s-s_1)^3(s-s_4)\ldots(s-s_n)}$$

$$F(s)=\frac{K_{11}}{(s-s_1)^3}+\frac{K_{12}}{(s-s_1)^2}+\frac{K_{13}}{(s-s_1)}+\sum_{k=4}^{n}\frac{K_k}{s-s_k}$$

$$(s-s_1)^3\frac{P(s)}{Q(s)}=K_{11}+(s-s_1)K_{12}+(s-s_1)^2K_{13}+(s-s_1)^3\sum_{k=4}^{n}\frac{K_k}{s-s_k}$$

$$K_{11}=\left[(s-s_1)^3\frac{P(s)}{Q(s)}\right]_{s=s_1}$$

$$K_{12}=\left[\frac{d}{ds}(s-s_1)^3\frac{P(s)}{Q(s)}\right]_{s=s_1}$$

$$K_{13}=\frac{1}{2}\left[\frac{d^2}{ds^2}(s-s_1)^3\frac{P(s)}{Q(s)}\right]_{s=s_1}$$

$$K_{rm}=\frac{1}{(m-1)!}\left[\frac{d^{m-1}}{ds^{m-1}}(s-s_r)^p\frac{P(s)}{Q(s)}\right]_{s=s_r}$$

$$m=1,2,\ldots p$$

$$f(t)=\frac{K_{11}}{2}t^2e^{s_1t}+K_{12}te^{s_1t}+K_{13}e^{s_1t}+\sum_{k=4}^{n}K_k\cdot e^{s_kt},\quad t>0$$

Polovi funkcije su konjugovano kompleksni

$$F(s)=\frac{P(s)}{Q(s)}=\frac{P(s)}{(s-s_1)(s-s_1^*)(s-s_3)\ldots(s-s_n)}$$

$$F(s)=\frac{K_1}{s-s_1}+\frac{K_1^*}{s-s_1^*}+\frac{K_3}{s-s_3}+\ldots+\frac{K_n}{s-s_n}$$

$$s_1=-\alpha+j\omega,\quad s_1^*=-\alpha-j\omega$$

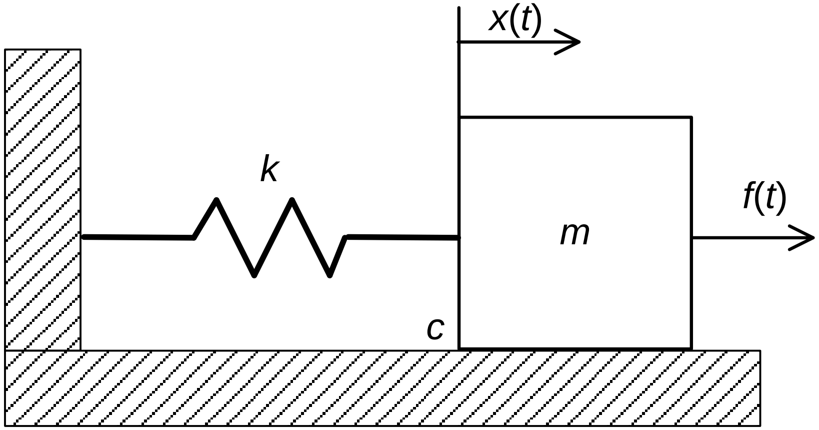
$$K_1=a+jb,\quad K_1^*=a-jb$$

$$K_1=a+jb=\left.\frac{P(s)}{Q'(s)}\right|_{s=-\alpha+j\omega}$$

$$F(s)=\frac{a+jb}{(s+\alpha)-j\omega}+\frac{a-jb}{(s+\alpha)+j\omega}+\sum_{k=3}^{n}\frac{K_k}{s-s_k}=\frac{2a(s+\alpha)-2b\omega}{(s+\alpha)^2+\omega^2}+\sum_{k=3}^{n}\frac{K_k}{s-s_k}$$

$$f(t)=2a\cdot e^{-\alpha t}\cos\omega t-2b\cdot e^{-\alpha t}\sin\omega t+\sum_{k=3}^{n}K_k e^{s_k t},\quad t>0$$

Primer primene Laplasove transformacije



- Mehanički sistem je opisan diferencijalnom jednačinom

$$m\frac{d^2x(t)}{dt^2}+c\frac{dx(t)}{dt}+k\cdot x(t)=f(t)$$

$$f(t)=0,\quad x(0_-)=x_0=1,\quad \frac{dx}{dt}(0_-)=0$$

Primer – nastavak

- Primena Laplasove transformacije na dif. jedn. daje:

$$m\left(s^2X(s) - sx(0_-) - \frac{dx}{dt}(0_-)\right) + c\left(sX(s) - x(0_-)\right) + k \cdot X(s) = 0$$

- a nakon sređivanja:

$$\left(ms^2 + cs + k\right)X(s) = \left(ms + c\right)x(0_-)$$

$$X(s) = \frac{ms + c}{ms^2 + cs + k}\,x(0_-) = \frac{ms + c}{ms^2 + cs + k}$$

$$X(s) = \frac{P(s)}{Q(s)} = \frac{s + \frac{c}{m}}{s^2 + \frac{c}{m}s + \frac{k}{m}}$$

Primer – realni jednostruki polovi

- Za *k*/*m*=2 i *c*/*m*=3 Razvoj u sumu parcijalnih sabiraka

$$X(s)=\frac{s+3}{s^2+3s+2}=\frac{s+3}{(s+1)(s+2)}=\frac{2}{s+1}-\frac{1}{s+2}$$

- Original x(t) se dobija promenim inverzne Laplasove transformacije (upotrebom tablica)

$$x(t)=L^{-1}\{X(s)\}=L^{-1}\left\{\frac{2}{s+1}\right\}-L^{-1}\left\{\frac{1}{s+2}\right\}=2e^{-t}-e^{-2t}$$

```
K=2; M=1; c=3;
P=[M c]; Q=[M c K];
[nule,polovi,ostatak]=residue(P,Q)

plot(polovi+eps*j,'x')

roots(Q)
```

```
nule =
    -1
     2
polovi =
    -2
    -1
ostatak =
     []
```

Primer – realni višestruki polovi

- Za *k*/*m*=4 i *c*/*m*=4 Razvoj u sumu parcijalnih sabiraka

$$X(s) = \frac{s+4}{s^2+4s+4} = \frac{s+4}{(s+2)^2} = \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

- Original x(t) se dobija primenom inverzne Laplasove transformacije (upotrebom tablica)

$$x(t) = L^{-1}\{X(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\} + L^{-1}\left\{\frac{2}{(s+2)^2}\right\} = e^{-2t} + 2te^{-2t}$$

```
K=4; M=1; c=4;
P=[M c]; Q=[M c K];
[nule,polovi,ostatak]=residue(P,Q)

plot(polovi+eps*j,'x')

roots(Q)
```

```
nule =
     1
     2
polovi =
    -2
    -2
ostatak =
     []
```

Primer – konjugovano-kompleksni polovi

- Za *k*/*m*=3 i *c*/*m*=2 Razvoj u sumu parcijalnih sabiraka

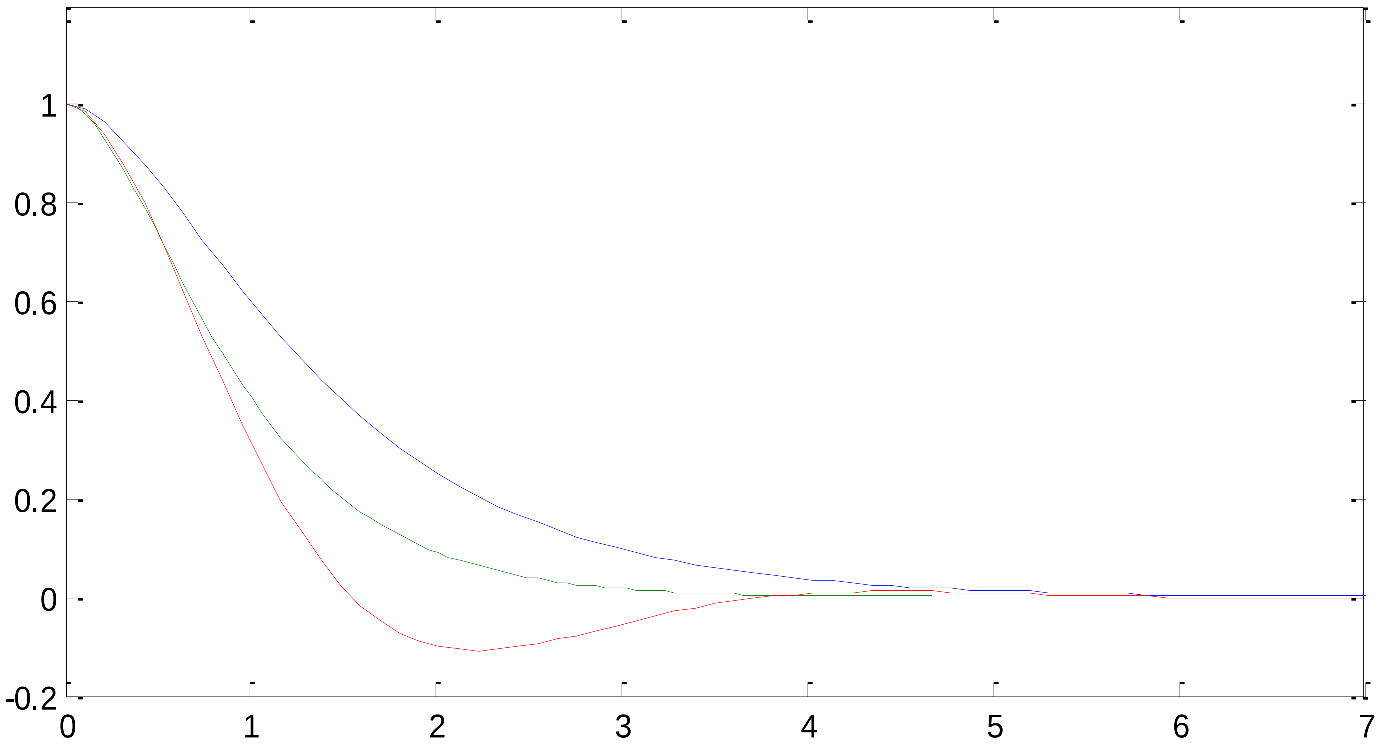
$$X(s) = \frac{s+2}{s^2+2s+3} = \frac{s+1+1}{(s+1)^2+(\sqrt{2})^2} = \frac{s+1}{(s+1)^2+(\sqrt{2})^2} + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}$$

- Original x(t) se dobija promenim inverzne Laplasove transformacije (upotrebom tablica)

$$x(t) = L^{-1}\{X(s)\} = L^{-1}\left\{\frac{s+1}{(s+1)^2+(\sqrt{2})^2}\right\} + \frac{1}{\sqrt{2}} \cdot L^{-1}\left\{\frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}\right\}$$

$$x(t) = e^{-t}\cos(t\sqrt{2}) + \frac{1}{\sqrt{2}}e^{-t}\sin(t\sqrt{2})$$

Primer – uporedni prikaz



Odziv modela 2. reda

- Uvođenjem smena model se može napisati kao

$$\xi = \frac{c}{2\sqrt{km}}$$ ← Faktor relativnog prigušenja

$$\omega_n = \sqrt{\frac{k}{m}}$$ ← Prirodna učestanost

$$X(s) = \frac{ms + c}{ms^2 + cs + k} x_0 = \frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2} x_0$$

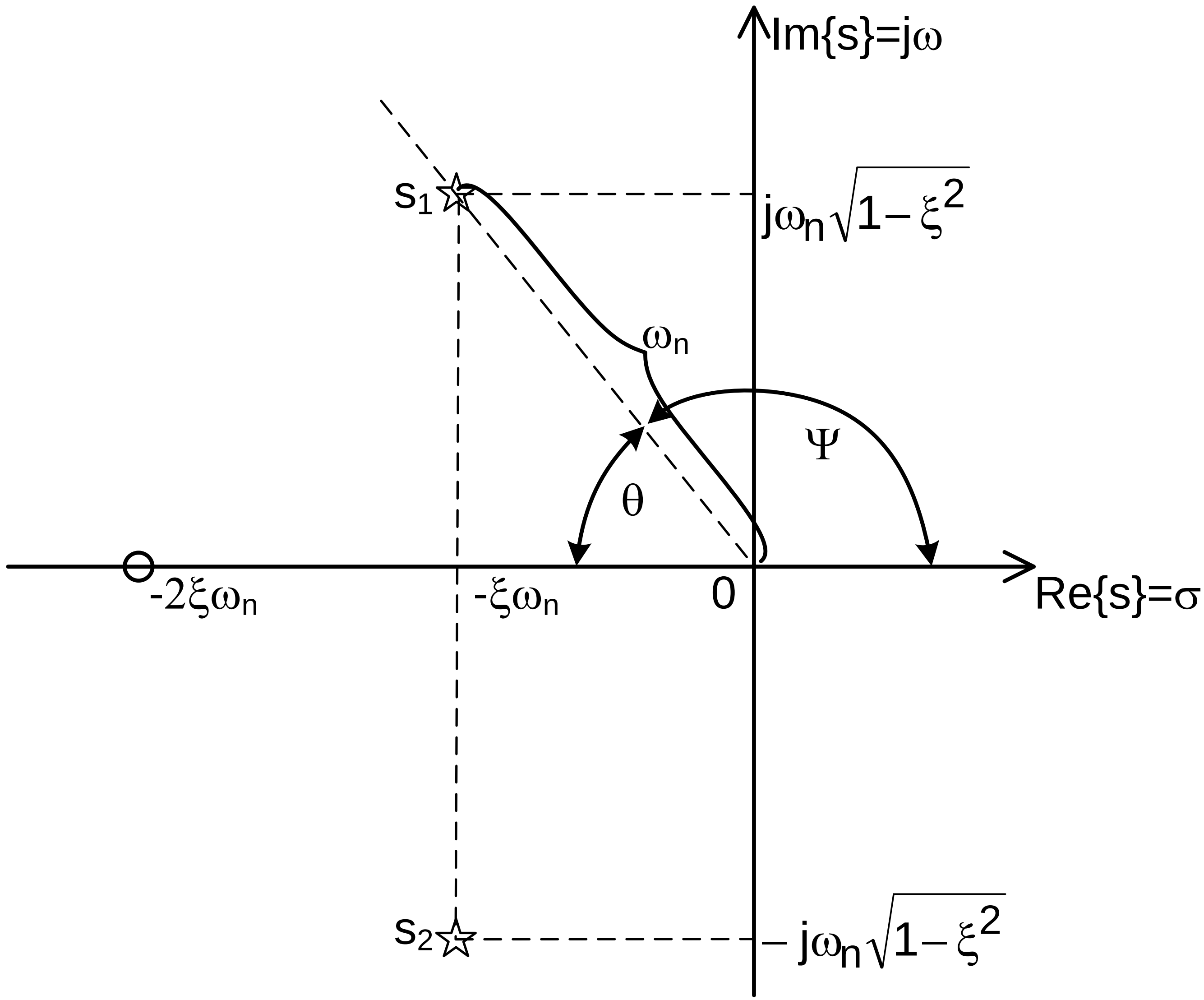
- Na karakter odziv sistema utiču polovi sistema

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

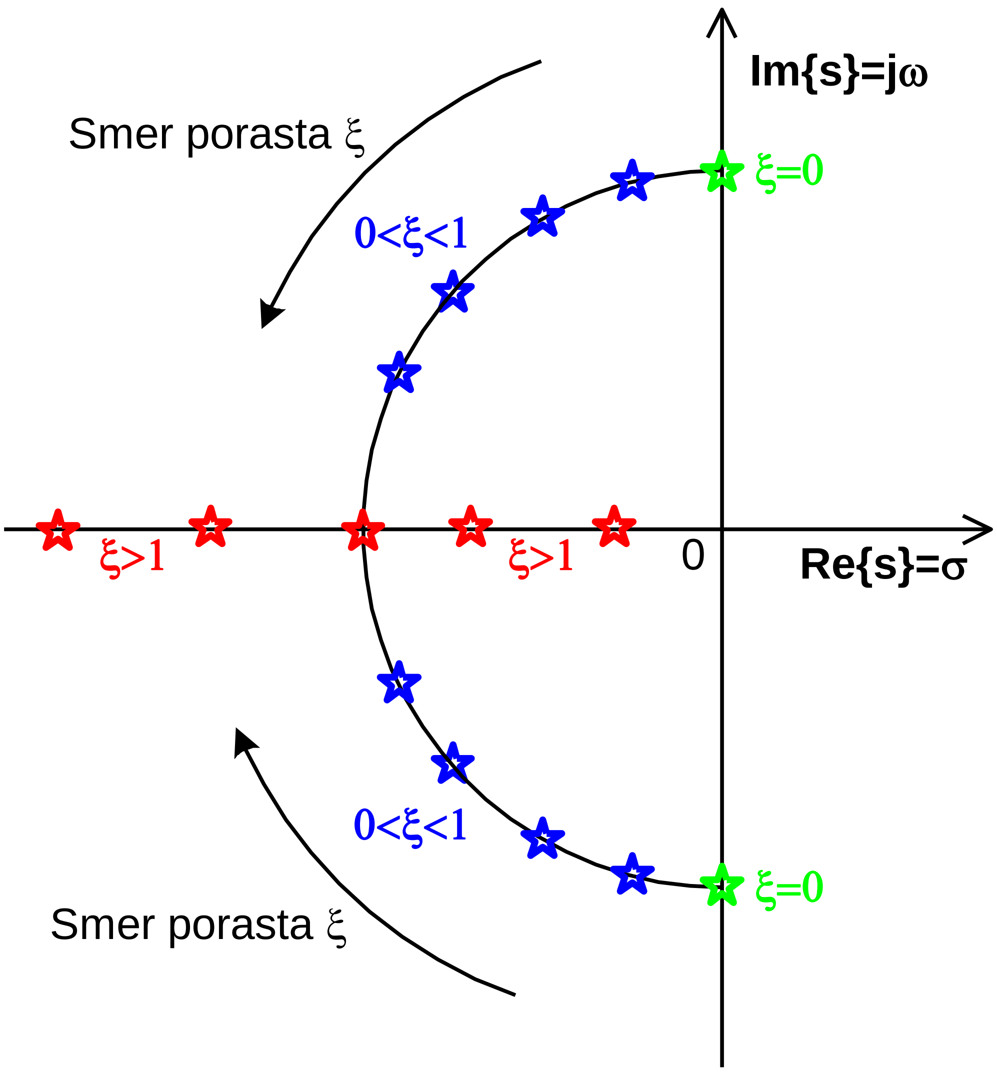
$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$ ← Realni koreni $\xi \geq 1$

$$s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1 - \xi^2}$$ ← Konjugovano-kompleksni polovi $\xi < 1$

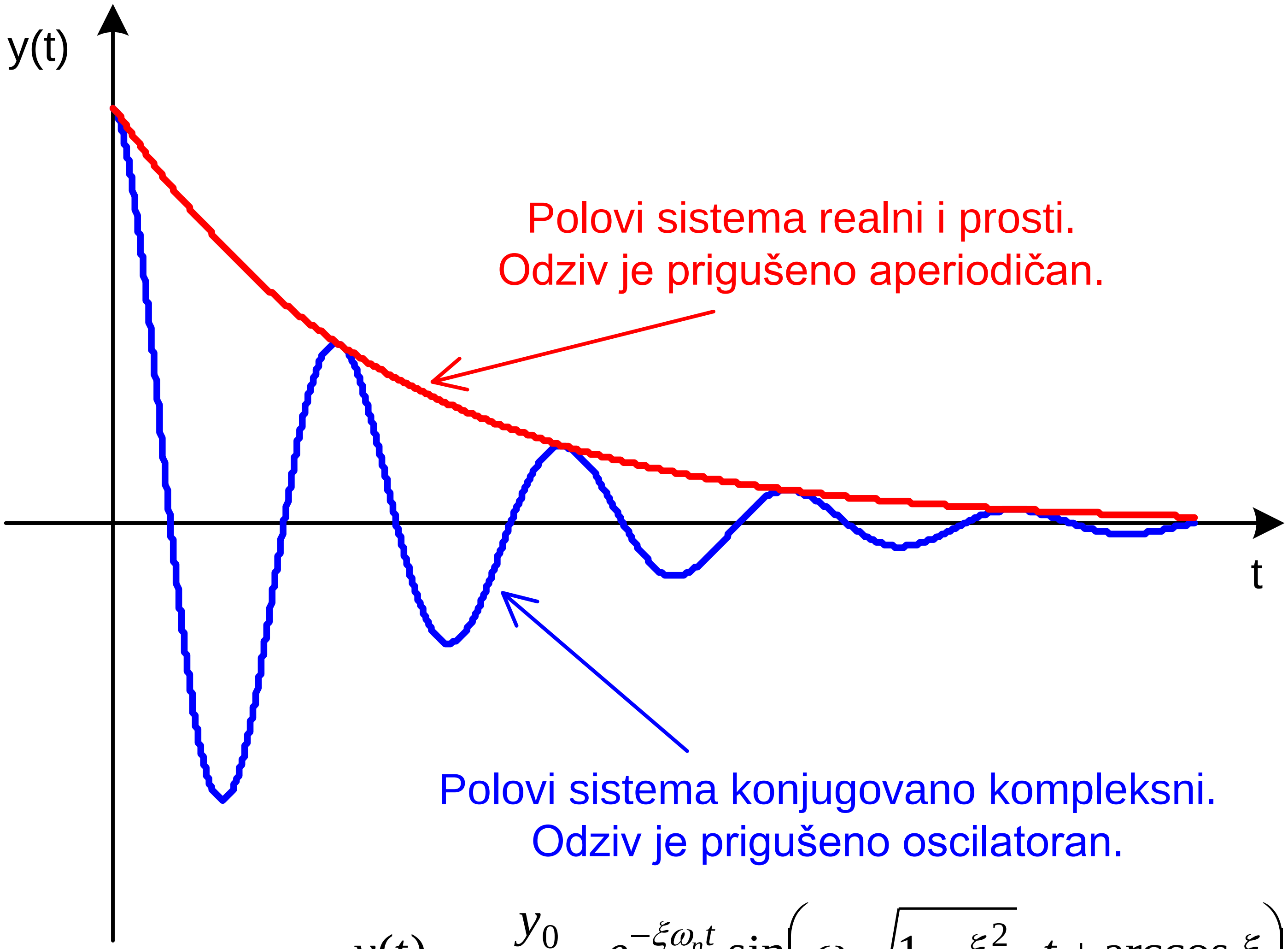
Lokacije konjugovano-kompleksnih polova



Uticaj ξ na lokacije polova



Prigušen odziv sistema



$$y(t) = \frac{y_0}{1-\xi^2} e^{-\xi\omega_n t} \sin\left(\omega_n \sqrt{1-\xi^2} \cdot t + \arccos \xi \right)$$

Primer

- Poznato

$$\dddot{x}(t) + 3\ddot{x}(t) + 5\dot{x}(t) + 3x(t) = 3 + 2\sin 2t$$
$$x(0) = \dot{x}(0) = \ddot{x}(0) = 0$$

- Rešenje

$$f(t) = 3 + 2\sin 2t$$
$$\dddot{x}(t) + 3\ddot{x}(t) + 5\dot{x}(t) + 3x(t) = f(t)$$

$$s^3X(s) + 3s^2X(s) + 5sX(s) + 3X(s) = F(s)$$
$$X(s) = \frac{1}{s^3 + 3s^2 + 5s + 3}F(s)$$
$$F(s) = \frac{3}{s} + \frac{4}{s^2 + 2^2}$$

…

$$X(s) = \frac{3s^2 + 4s + 12}{s^6 + 3s^5 + 9s^4 + 15s^3 + 20s^2 + 12s}$$

$$X(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \frac{r_3}{s - p_3} + \frac{r_4}{s - p_4} + \frac{r_5}{s - p_5} + \frac{r_6}{s - p_6}$$

$$r_1 = -0.0235 + 0.1059j, p_1 = 2j$$
$$r_2 = r_1^*, p_2 = p_1^*$$
$$r_3 = 0.0735 + 0.1872j, p_3 = -1 + j\sqrt{2}$$
$$r_4 = r_3^*, p_4 = p_3^*$$
$$r_5 = -1.1, p_5 = -1$$
$$r_6 = 1, p_6 = 0$$

$$X(s) = \frac{-0.0471s - 0.4235}{s^2 + 4} + \frac{0.1471s - 0.3824}{s^2 + 2s + 3} + \frac{-1.1}{s + 1} + \frac{1}{s}$$

$$X(s) = -0.0471\frac{s}{s^2+2^2} - \frac{0.4235}{2}\frac{2}{s^2+2^2} +$$
$$0.1471\frac{s+1}{(s+1)^2+\left(\sqrt{2}\right)^2} + \frac{0.1471-0.3824}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+\left(\sqrt{2}\right)^2} + \frac{-1.1}{s+1} + \frac{1}{s}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

$$x(t) = -0.0471\cos 2t - 0.2117\sin 2t +$$
$$0.1471e^{-t}\cos t\sqrt{2} - 0.1664e^{-t}\sin t\sqrt{2} - 1.1e^{-t} + 1$$

Funkcija prenosa

- Obična linearna diferencijalna jednačina modela 1 ulaz-1 izlaz

$$y^{(n)} + a_{n-1}y^{(n-1)}(t) + \cdots + a_2\ddot{y}(t) + a_1\dot{y}(t) + a_0y(t)$$
$$= b_m u^{(m)}(t) + b_{m-1}u^{(m-1)}(t) + \cdots + b_1\dot{u}(t) + b_0u(t) \qquad n \geq m$$

- Primeni se Laplasova transformacija
  - model se nalazi u radnoj tački (sve početne vrednosti su 0)

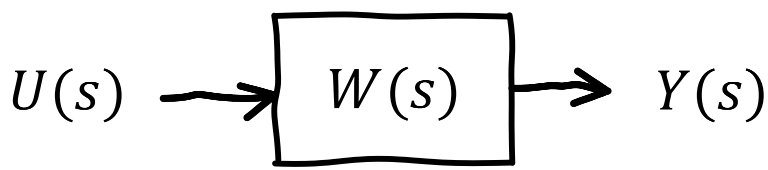
$$s^nY(s) + a_{n-1}s^{n-1}Y(s) + \cdots + a_2s^2Y(s) + a_1sY(s) + a_0Y(s)$$
$$= b_ms^mU(s) + b_{m-1}s^{m-1}U(s) + \cdots + b_1sU(s) + b_0U(s)$$

- Dobije se funkcija prenosa

$$\frac{Y(s)}{U(s)} = W(s) = \frac{b_ms^m + b_{m-1}s^{m-1} + \cdots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_2s^2 + a_1s + a_0}.$$

Funkcija prenosa (2)

- Definicija:
  Funkcija prenosa je odnos Laplasove transformacije izlaznog ($Y(s) = \mathcal{L}\{y(t)\}$) i ulaznog ($U(s) = \mathcal{L}\{u(t)\}$) signala, uz pretpostavku da su svi početni uslovi jednaki nuli i da je $u(t) = y(t) \equiv 0$ za svako $t < 0$.
- uzima samo odnos ulaz-izlaz
- ne pruža nikakvu informaciju o unutrašnjoj strukturi i ponašanju sistema



- Predstavlja se količnikom polinoma

$$W(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0} = \frac{P(s)}{Q(s)}.$$

Funkcija prenosa – faktorizovan oblik

$$W(s) = k\frac{(s-r_1)(s-r_2)\dots(s-r_m)}{(s-t_1)(s-t_2)\dots(s-t_n)}$$

- k – pojačanje ($k = b_m$)
- $r_1, r_2, \dots, r_m$ su nule sistema tj. koreni polinoma brojioca
- $t_1, t_2, \dots, t_n$ su polovi sistema

Funkcija prenosa multivarijabilnog sistema

- Posmatraju se svi parovi ulaz-izlaz

$$G_{kj}(s) = \frac{Y_k(s)}{U_j(s)}, \qquad k = 1,2,\dots,r, \qquad j = 1,2,\dots,m.$$

- Nakon primene principa superpozicije

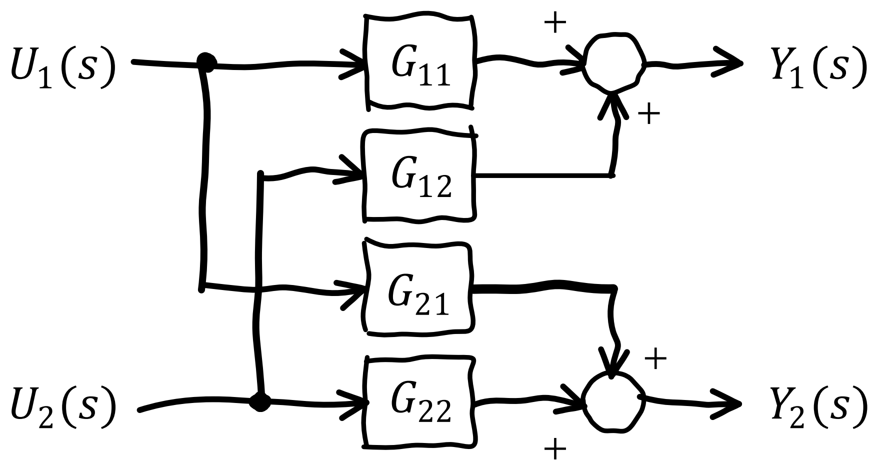
$$Y_k(s) = G_{k1}(s)U_1(s) + G_{k2}(s)U_2(s) + \cdots + G_{km}(s)U_m(s), \qquad k = 1,2,\dots,r$$

- Vektorska notacija

$$\boldsymbol{Y}(s) = \boldsymbol{G}(s) \cdot \boldsymbol{U}(s)$$

$$\boldsymbol{G}(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \dots & G_{1m}(s) \\ G_{21}(s) & G_{22}(s) & \dots & G_{2m}(s) \\ \dots & \dots & & \dots \\ G_{r1}(s) & G_{r2}(s) & \dots & G_{rm}(s) \end{bmatrix}, \boldsymbol{U}(s) = \begin{bmatrix} U_1(s) \\ U_2(s) \\ \dots \\ U_m(s) \end{bmatrix}, \boldsymbol{Y}(s) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \dots \\ Y_r(s) \end{bmatrix}.$$

Primer funkcije prenosa multivarijabilnog sistema



$$Y_1(s) = G_{11}(s)U_1(s) + G_{12}(s)U_2(s)$$
$$Y_2(s) = G_{21}(s)U_1(s) + G_{22}(s)U_2(s)$$

$$\boldsymbol{G}(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$$

Dobijanje funkcije prenosa od linearnog matematičkog modela u prostoru stanja

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{A} \cdot \hat{\boldsymbol{x}}(t) + \boldsymbol{B} \cdot \hat{\boldsymbol{u}}(t), \qquad \hat{\boldsymbol{x}}(0) = \boldsymbol{x}_0$$

$$\hat{\boldsymbol{y}}(t) = \boldsymbol{C} \cdot \hat{\boldsymbol{x}}(t) + \boldsymbol{D} \cdot \hat{\boldsymbol{u}}(t)$$

⇦ Primeni se Laplasova transformacija

$$s\boldsymbol{X}(s) = \boldsymbol{A} \cdot \boldsymbol{X}(s) + \boldsymbol{B} \cdot U(s)$$

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1} \cdot \boldsymbol{B} \cdot U(s)$$

$$Y(s) = \boldsymbol{C} \cdot \boldsymbol{X}(s) + \boldsymbol{D} \cdot U(s)$$

$$Y(s) = (\boldsymbol{C} \cdot (s\boldsymbol{I} - \boldsymbol{A})^{-1} \cdot \boldsymbol{B} + \boldsymbol{D})U(s)$$

$$\frac{Y(s)}{U(s)} = W(s) = \boldsymbol{C} \cdot (s\boldsymbol{I} - \boldsymbol{A})^{-1} \cdot \boldsymbol{B} + \boldsymbol{D}.$$

Primer

- Dobijanje funkcije prenosa na osnovu linearnog matematičkog modela iz zadatka sa električnim kolom …

Prednosti koncepta prostora stanja

- Koncept prostora stanja ima nekoliko prednosti u odnosu na klasični pristup (kompleksan domen), posebno ako se posmatra sa aspekta korišćenja digitalnih računara:
  - Određivanje rešenja sistema diferencijalnih jednačina prvog reda je brže na digitalnom računaru, nego rešavanje odgovarajuće diferencijalne jednačine višeg reda.
  - Uprošteno je matematičko opisivanje upotrebom vektorske notacije
  - Uključivanje početnih uslova sistema je jednostavno.
  - Model se može primeniti na vremenski promenljive, nelinearne, stohastičke i diskretne sisteme.

Analiza ponašanja modela u prostoru stanja

- U vremenskom domenu

$$\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$$
$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}(t) + \mathbf{D} \cdot \mathbf{u}(t)$$

- Nakon primene Laplace-ove transformacije dobija se:
  Početni uslovi ne moraju biti nula!

$$s\mathbf{X}(s) - \mathbf{X}(0_-) = \mathbf{A} \cdot \mathbf{X}(s) + \mathbf{B} \cdot U(s)$$
$$(s\mathbf{I} - \mathbf{A}) \cdot \mathbf{X}(s) = \mathbf{X}(0_-) + \mathbf{B} \cdot U(s)$$
$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{X}(0_-) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \cdot U(s)$$

$$Y(s) = \mathbf{C} \cdot \mathbf{X}(s) + \mathbf{D} \cdot U(s)$$
$$Y(s) = \mathbf{C} \cdot (s\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{X}(0_-) + \left(\mathbf{C} \cdot (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right) \cdot U(s)$$

- Primena inverzne Laplasove transformacije na prethodni izraz daje kretanje izlaza modela ...

Kretanje modela

- Računa se preko Fundamentalne matrice sistema $\boldsymbol{\Phi}(t) = L^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\}$

- Kretanje promenljivih stanja **x**(*t*)

$$\mathbf{x}(t) = \boldsymbol{\Phi}(t) \cdot \mathbf{x}(0) + \int_0^t \boldsymbol{\Phi}(t-\tau)\mathbf{B}u(\tau)d\tau$$

kretanje stanja sistema pod dejstvom spoljne pobude

kretanje stanja sistema pod dejstvom početnih uslova

- Kretanje izlaza **y**(*t*)

$$y(t) = \mathbf{C} \cdot \boldsymbol{\Phi}(t) \cdot \mathbf{x}(0) + \mathbf{C} \cdot \int_0^t \boldsymbol{\Phi}(t-\tau)\mathbf{B}u(\tau)d\tau + \mathbf{D} \cdot u(t)$$

kretanje izlaza sistema pod dejstvom spoljne pobude

kretanje izlaza sistema pod dejstvom početnih uslova