UNIVERZITET U NOVOM SADU
FAKULTET TEHNIČKIH NAUKA
KATEDRA ZA AUTOMATIKU I UPRAVLJANJE SISTEMIMA

Vremenski diskretni modeli

Modeliranje i simulacija sistema

Upravljanje, modelovanje i simulacija sistema

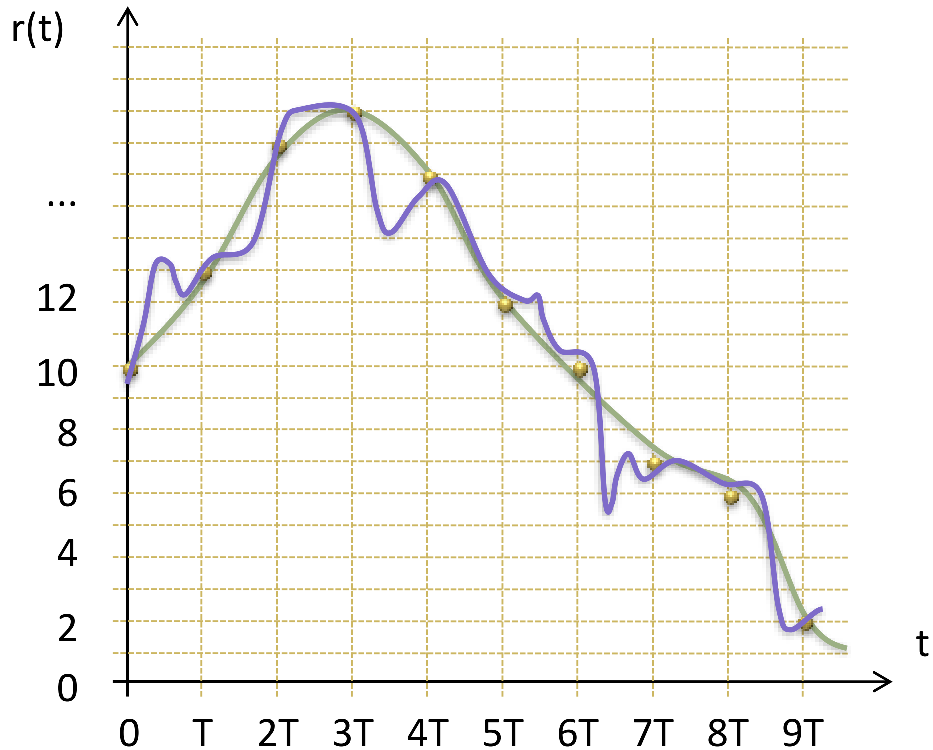
Digitalan model

- Razlozi upotrebe digitalnih modela
  - Upotreba računara (obrada)
  - Kodovanje signala
    - Prenos signala na daljinu bez smetnji
  - Vremenski multipleks
- Ovde posmatramo formiranje digitalnog (vremenski diskretnog) linearnog modela u prostoru stanja na osnovu vremenski kontinualnog linearnog modela u prostoru stanja

Kvantovanje signala

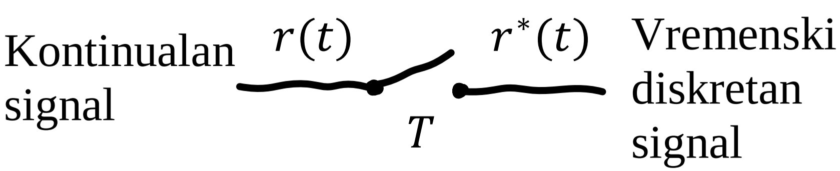
- Signali se predstavljaju brojnim vrednostima
- Signali se diskretizuju
  - Po vremenu
  - Po nivou (kvantovanje po nivou)
- Kvantovanje po vremenu vrši „odabirač“
  - Na izlazu se pojavljuje povorka impulsa (odbiraka) u trenucima odabiranja $t=kT$, $k=0,1,2,...$
  - Perioda odabiranja T
- Kvantovanje po nivou vrši A/D konvertor (analogno-digitalni)
  - Na izlazu se dobijaju brojne vrednosti
  - Broj nivoa (kvantova) zavisi od rezolucije A/D konvertora
    - 2^8, 2^{10}, 2^{12}, 2^{16}
  - Mi ćemo pretpostaviti da je rezolucija A/D konvertora dovoljno velika

Kvantovanje signala - primer

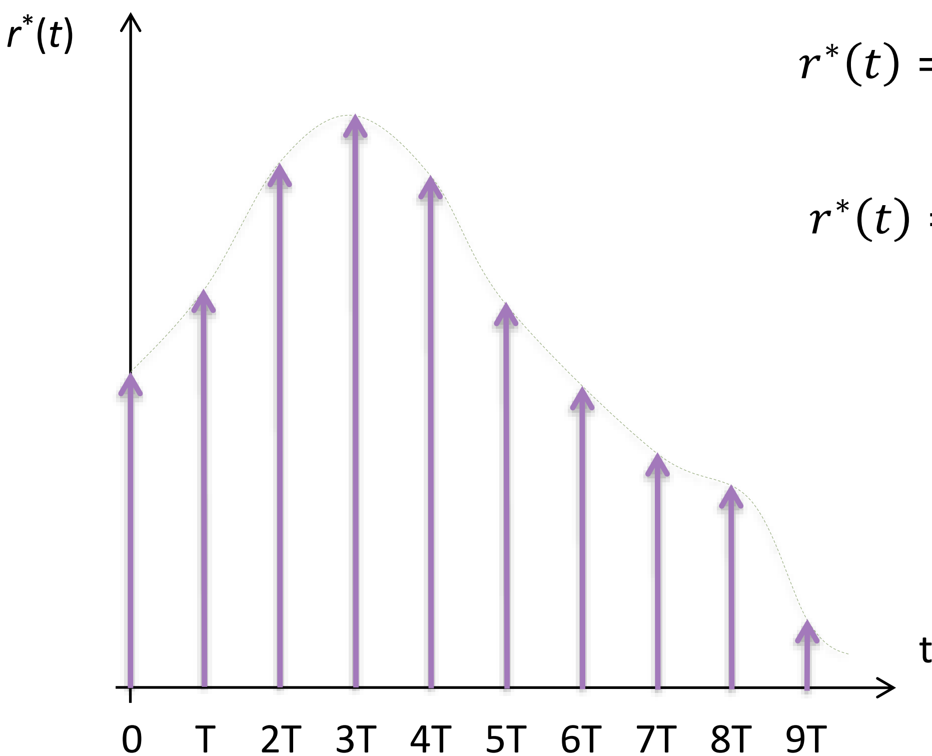


r* = [10 13 17 18 16 12 10 7 6 2]

Odabirač



$$r^*(t) = \begin{cases} r(kT), & t = kT \\ 0, & t \neq kT \end{cases} \quad k = 0,1,\ldots$$

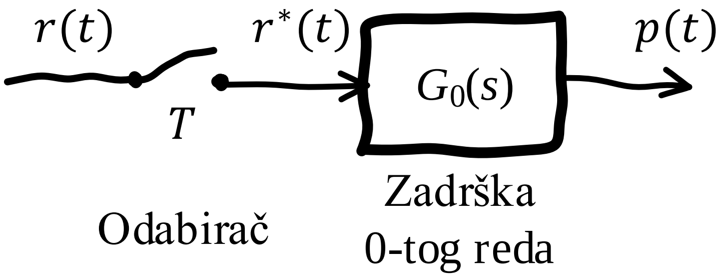


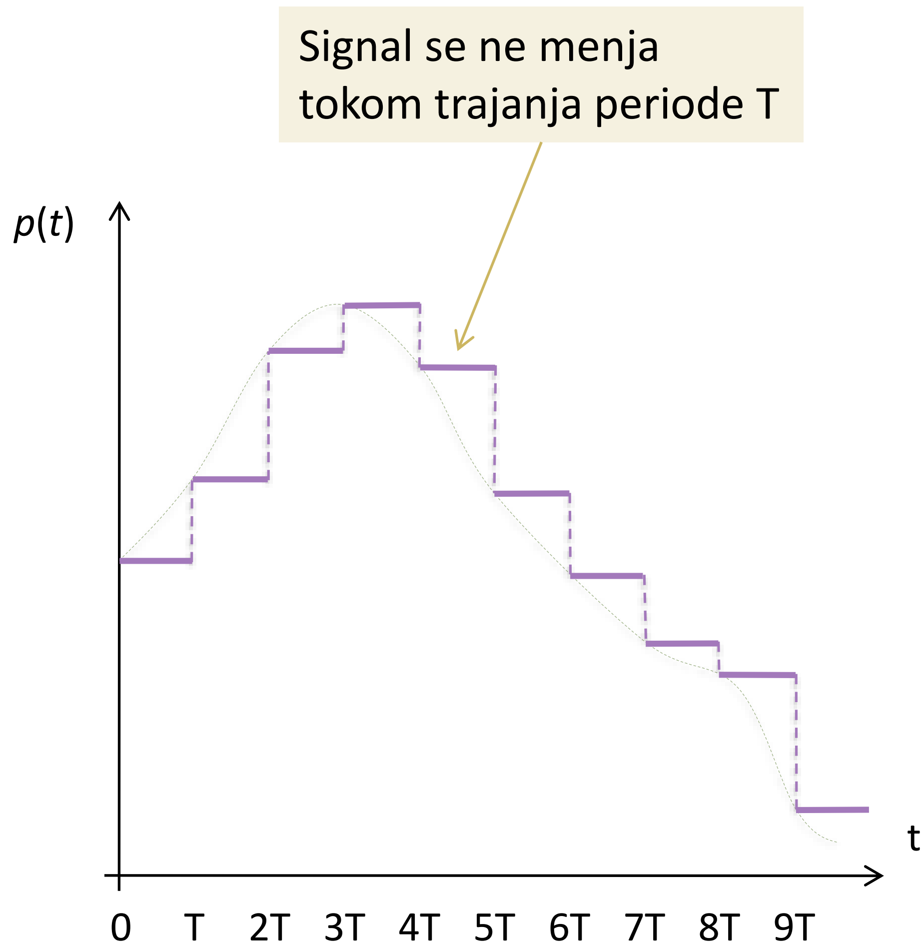
$$r^*(t) = r(kT) \cdot \delta(t - kT), \qquad k = 0,1,\ldots$$

$$r^*(t) = \sum_{k=0}^{\infty} r(kT) \cdot \delta(t - kT)$$

Kolo zadrške nultog reda

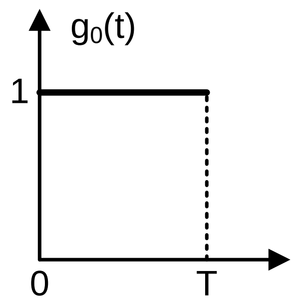
- Realizacija procesa odabiranja vrži i zadršku signala





Model zadrške nultog reda

Jedan odbirak:



$$g_0(t) = h(t) - h(t-T)$$

$$G_0(s) = \frac{1}{s} - \frac{e^{-sT}}{s} = \frac{1-e^{-sT}}{s}$$

Model zadrške

Ceo diskretizovan signal:

$$p(t) = \sum_{k=0}^{\infty} r(kT) \cdot \{h(t-kT) - h(t-(k+1)T)\} \qquad / L$$

$$P(s) = \sum_{k=0}^{\infty} r(kT) \cdot \left\{ \frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s} \right\}$$

$$P(s) = \sum_{k=0}^{\infty} r(kT) \cdot \frac{1-e^{-sT}}{s} \cdot e^{-kTs} = \frac{1-e^{-sT}}{s} \cdot \sum_{k=0}^{\infty} r(kT) \cdot e^{-kTs}$$

$$P(s) = G_0(s) \cdot R^*(s)$$

$$R^*(s) = \sum_{k=0}^{\infty} r(kT) \cdot e^{-kTs}$$

Kompleksan lik povorke odbiraka r*(t)

Z-transformacija

- Povorka vremenski diskretizovanog signala i njegova Laplasova transformacija

$$r^*(t) = \sum_{k=0}^{\infty} r(kT) \cdot \delta(t - kT)$$

$$R^*(s) = \mathcal{L}\{r^*(t)\} = \sum_{k=0}^{\infty} r(kT) \cdot \mathcal{L}\{\delta(t - kT)\} = \sum_{k=0}^{\infty} r(kT) \cdot e^{-kTs}$$

- Smena uvodi novu kompleksnu promenljivu z

$$z = e^{sT}$$

- Z-transformacija povorke signala

$$\mathcal{Z}\{r(t)\} = \mathcal{Z}\{r^*(t)\} = R(z)$$

$$R(z) = \sum_{k=0}^{\infty} r(kT) \cdot z^{-k}$$

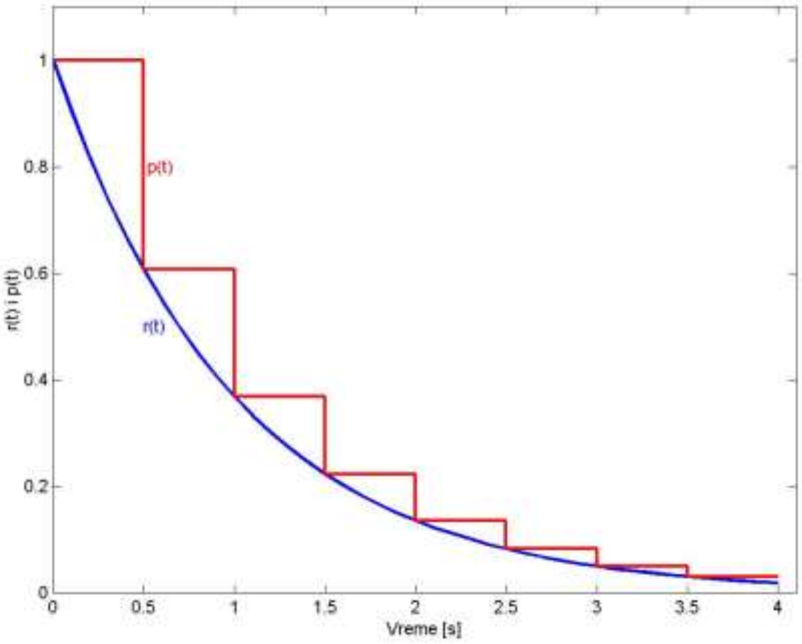
Teorema o odabiranju

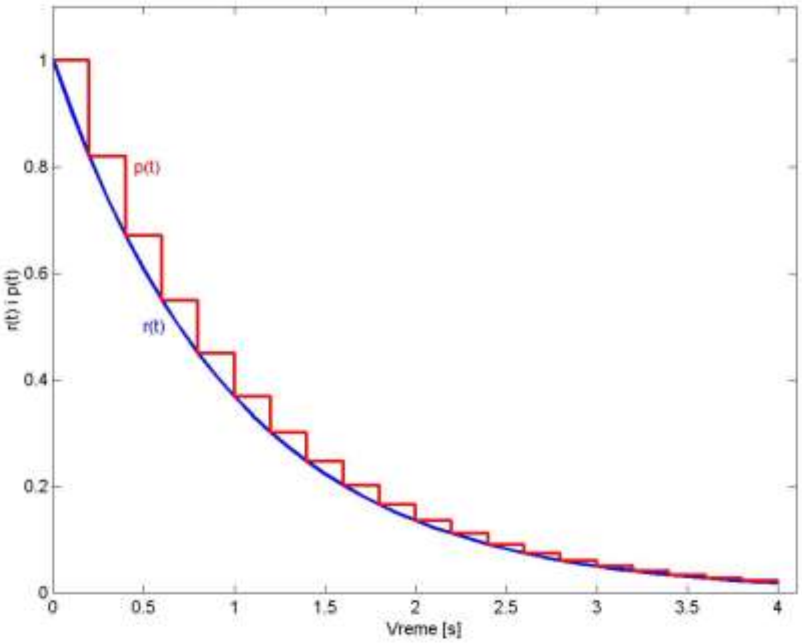
- Ako kontinualan signal $f(t)$ ne sadrži harmonike u području učestanosti $>$ ω_0 [rad/s], on se može kompletno okarakterisati vrednostima signala merenim u trenucima međusobno udaljenim za vreme $T = 0.5(2\pi/\omega 0)$.

- Teorijski: Perioda uzorkovanje signala treba da je barem 2 puta kraća od periode (prostoperiodične) komponente signala koja ima najveću učestanost
- Praktično: Perioda uzorkovanja treba da je 10-20 puta kraća

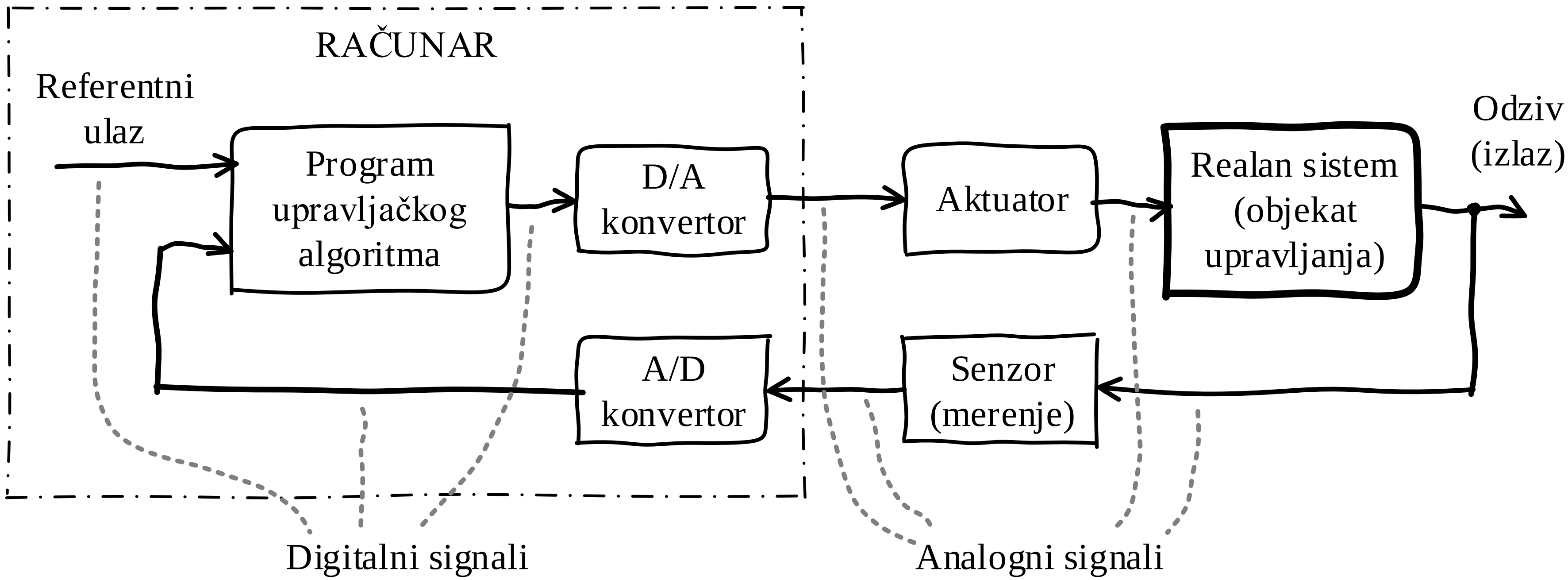
Primer odabiranja

- Primer signala odabiranog različitim periodama odabiranja
  - T_a=0.5s
  - T_b=0.2s



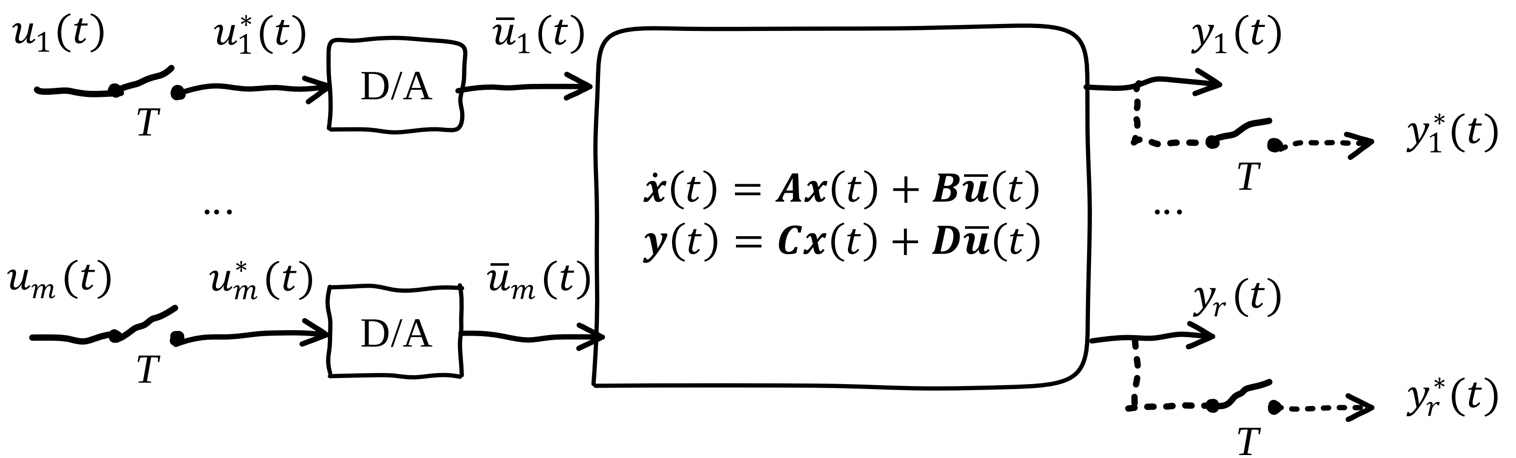


Upravljanje računarom



Odziv na diskretnu pobudu

- Tokom jedne periode T ulazi $u(t)$ su konstantni
- Izlazi sistema nas intresuju samo u trenucima odabiranja $t = kT$



$$\boldsymbol{x}(k+1) = \boldsymbol{E}\cdot\boldsymbol{x}(k) + \boldsymbol{F}\cdot\boldsymbol{u}(k), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$
$$\boldsymbol{y}(k) = \boldsymbol{C}\cdot\boldsymbol{x}(k) + \boldsymbol{D}\cdot\boldsymbol{u}(k)$$

$$k \equiv kT, \qquad k = 0,1,2,\dots$$

Linearni modeli u prostoru stanja

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A} \cdot \boldsymbol{x}(t) + \boldsymbol{F} \cdot \boldsymbol{u}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$
$$\boldsymbol{y}(t) = \boldsymbol{C} \cdot \boldsymbol{x}(t) + \boldsymbol{D} \cdot \boldsymbol{u}(t)$$

Linearan **kontinualan** matematički model u prostoru stanja

$$\boldsymbol{x}(k+1) = \boldsymbol{E} \cdot \boldsymbol{x}(k) + \boldsymbol{F} \cdot \boldsymbol{u}(k), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$
$$\boldsymbol{y}(k) = \boldsymbol{C} \cdot \boldsymbol{x}(k) + \boldsymbol{D} \cdot \boldsymbol{u}(k)$$

$$k \equiv kT, \qquad k = 0,1,2,\ldots$$

Linearan **vremenski diskretan** matematički model u prostoru stanja

Računanje diskretanog odziva

- Vrednosti promenljivih stanja u diskretnim trenucima se mogu izračunati rekurzivno na osnovu poznavanja $x(0)$ i vrednosti ulaza u svim trenucima $u(k), k = 0,1,2, \ldots$

$$\boldsymbol{x}(1) = \boldsymbol{E} \cdot \boldsymbol{x}(0) + \boldsymbol{F} \cdot \boldsymbol{u}(0)$$
$$\boldsymbol{x}(2) = \boldsymbol{E} \cdot \boldsymbol{x}(1) + \boldsymbol{F} \cdot \boldsymbol{u}(1)$$
$$\ldots$$
$$\boldsymbol{x}(k+1) = \boldsymbol{E} \cdot \boldsymbol{x}(k) + \boldsymbol{F} \cdot \boldsymbol{u}(k)$$

Kontinualan odziv

- Posmatramo linearan matematički model u prostoru stanja

$$\dot{\mathbf{x}}(t) = \mathbf{A}\cdot\mathbf{x}(t) + \mathbf{B}\cdot\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \qquad (*)$$
$$\mathbf{y}(t) = \mathbf{C}\cdot\mathbf{x}(t) + \mathbf{D}\cdot\mathbf{u}(t)$$

Linearan kontinualan matematički model u prostoru stanja

- Odziv modela je

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\cdot\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\cdot\mathbf{B}\cdot\mathbf{u}(\tau)d\tau, \quad 0 \le t \qquad (**)$$
$$\mathbf{y}(t) = \mathbf{C}\cdot\mathbf{x}(t) + \mathbf{D}\cdot\mathbf{u}(t)$$

gde je $\mathbf{\Phi}(t)$ fundamentalna matrica sistema $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$

- Izvođenje:
  - Nakon primene Laplasove transformacije na (*) dobija se

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\cdot\mathbf{X}(s) + \mathbf{B}\cdot\mathbf{U}(s)$$
$$s\mathbf{X}(s) - \mathbf{A}\cdot\mathbf{X}(s) = (s\mathbf{I}-\mathbf{A})\cdot\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\cdot\mathbf{U}(s)$$
$$\mathbf{X}(s) = (s\mathbf{I}-\mathbf{A})^{-1}\cdot\mathbf{x}(0) + (s\mathbf{I}-\mathbf{A})^{-1}\cdot\mathbf{B}\cdot\mathbf{U}(s)$$

  - Primenom inverzne Laplasove transformacije na X(s) dobija se (**)

$$\Phi(t) = L^{-1}\{(s\mathbf{I}-\mathbf{A})^{-1}\}$$

Primer računanja fundamentalne matrice

$$A=\begin{bmatrix}-3 & -2\\ 1 & 0\end{bmatrix}\quad B=\begin{bmatrix}1\\ 0\end{bmatrix}\quad C=\begin{bmatrix}0 & 5\end{bmatrix}\quad D=0$$

$$R(s)=(sI-A)^{-1}=\begin{bmatrix}s+3 & 2\\ -1 & s\end{bmatrix}^{-1}=\frac{1}{(s+1)(s+2)}\begin{bmatrix}s & -2\\ 1 & s+3\end{bmatrix}$$

$$R(s)=\begin{bmatrix}\frac{-1}{s+1}+\frac{2}{s+2} & \frac{-2}{s+1}+\frac{2}{s+2}\\ \frac{1}{s+1}+\frac{-1}{s+2} & \frac{2}{s+1}+\frac{-1}{s+2}\end{bmatrix}$$

$$\Phi(t)=L^{-1}\left\{R(s)\right\}=\begin{bmatrix}-e^{-t}+2e^{-2t} & -2e^{-t}+2e^{-2t}\\ e^{-t}-e^{-2t} & 2e^{-t}-e^{-2t}\end{bmatrix}$$

Vremenski diskretan odziv

- Na osnovu (**) odziv u trenutku $t=kT$ je

$$\mathbf{x}(kT) = \mathbf{\Phi}(kT)\cdot\mathbf{x}(0) + \int_0^{kT}\mathbf{\Phi}(kT-\tau)\cdot\mathbf{B}\cdot\mathbf{u}(\tau)d\tau$$

- a u narednom trenutku $t=kT+T$ je

$$\mathbf{x}(kT+T) = \mathbf{\Phi}(kT+T)\cdot\mathbf{x}(0) + \int_0^{kT}\mathbf{\Phi}(kT+T-\tau)\cdot\mathbf{B}\cdot\mathbf{u}(\tau)d\tau + \int_{kT}^{kT+T}\mathbf{\Phi}(kT+T-\tau)\cdot\mathbf{B}\cdot\mathbf{u}(\tau)d\tau$$

- Na osnovu osobine $\mathbf{\Phi}(a+b) = \mathbf{\Phi}(a)\cdot\mathbf{\Phi}(b)$

$$\mathbf{x}(kT+T) = \mathbf{\Phi}(T)\cdot\left(\mathbf{\Phi}(kT)\cdot\mathbf{x}(0) + \int_0^{kT}\mathbf{\Phi}(kT-\tau)\cdot\mathbf{B}\cdot\mathbf{u}(\tau)d\tau\right) + \int_{kT}^{kT+T}\mathbf{\Phi}(kT+T-\tau)\cdot\mathbf{B}\cdot\mathbf{u}(\tau)d\tau$$

- period $[kT, kT+T]$ je dovoljano kratak da možemo smatrati da je $\mathbf{u}(kT)$ nepromenljivo tokom tog perioda.

$$\mathbf{x}(kT+T) = \mathbf{\Phi}(T)\cdot\mathbf{x}(kT) + \left(\int_{kT}^{kT+T}\mathbf{\Phi}(kT+T-\tau)d\tau\right)\cdot\mathbf{B}\cdot\mathbf{u}(kT)$$

Vremenski diskretan odziv (2)

- Rešavanje se nastavlja uvođenjem smene $q = \tau - kT$

$$\int_{kT}^{kT+T} \mathbf{\Phi}(kT + T - \tau)d\tau = \int_{0}^{T} \mathbf{\Phi}(T-q)dq$$

i još jedna smena $y = T - q$

$$\int_{0}^{T} \mathbf{\Phi}(T-q)dq = -\int_{T}^{0} \mathbf{\Phi}(y)dy = \int_{0}^{T} \mathbf{\Phi}(y)dy$$

- Dobija se: $\mathbf{x}(kT+T) = \mathbf{\Phi}(T)\cdot\mathbf{x}(kT) + \left(\int_{0}^{T}\mathbf{\Phi}(t)dt\right)\cdot\mathbf{B}\cdot\mathbf{u}(kT)$

tj.

$$\boxed{\begin{aligned} &\mathbf{x}(k+1) = \mathbf{E}\cdot\mathbf{x}(k) + \mathbf{F}\cdot\mathbf{u}(k) \\ &\quad \mathbf{E} = \mathbf{\Phi}(T) \\ &\quad \mathbf{F} = \left(\int_{0}^{T}\mathbf{\Phi}(t)dt\right)\cdot\mathbf{B} \end{aligned}}$$

Primer: Linearan vremenski kontinualan matematički model u prostoru stanja

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \qquad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 0 & 5 \end{bmatrix} \boldsymbol{x}(t)$$

Matrice modela su

$$\boldsymbol{A} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} 0 & 5 \end{bmatrix}, \qquad \boldsymbol{D} = 0$$

Fundamentalna matrica

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} -e^{-t} + 2e^{-2t} & -2e^{-t} + 2e^{-2t} \\ e^{-t} - e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix}$$

Vremenska diskretizacija

$$\boldsymbol{E}(\mathrm{T}) = \boldsymbol{\Phi}(T) = \begin{bmatrix} -e^{-T} + 2e^{-2T} & -2e^{-T} + 2e^{-2T} \\ e^{-T} - e^{-2T} & 2e^{-T} - e^{-2T} \end{bmatrix}$$

$$\boldsymbol{F}(T) = \left(\int_0^T \boldsymbol{\Phi}(t) dt \right) \boldsymbol{B} = \begin{bmatrix} \int_0^T (-e^{-t} + 2e^{-2t}) dt \\ \int_0^T (e^{-t} - e^{-2t}) dt \end{bmatrix} = \begin{bmatrix} e^{-T} + e^{-2T} \\ \frac{1}{2} - e^{-T} + \frac{1}{2} e^{-2T} \end{bmatrix}$$

Za $T = 0.1$ dobija se:

$$\boldsymbol{E} = \begin{bmatrix} 0.7326 & -0.1722 \\ 0.0861 & 0.9909 \end{bmatrix}, \qquad \boldsymbol{F} = \begin{bmatrix} 0.0861 \\ 0.0045 \end{bmatrix}$$

Linearan vremenski diskretan model u prostoru stanja

$$\boldsymbol{x}(k+1) = \begin{bmatrix} 0.7326 & -0.1722 \\ 0.0861 & 0.9909 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 0.0861 \\ 0.0045 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 5 \end{bmatrix} \boldsymbol{x}(k)$$

Kako da izračunamo matrice E i F?

- Tražimo pogodnu transformaciju tako da model možemo opisati kao

$$\mathbf{x}(t) = \mathbf{P} \cdot \tilde{\mathbf{x}}(t)$$
$$\dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{A}} \cdot \tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}} \cdot \mathbf{u}(t)$$

- Ako je
$$\tilde{\mathbf{A}} = \begin{bmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n \end{bmatrix}$$
tada je
$$e^{\tilde{\mathbf{A}}} = \begin{bmatrix} e^{\lambda_1} & & & \mathbf{0} \\ & e^{\lambda_2} & & \\ & & \ddots & \\ \mathbf{0} & & & e^{\lambda_n} \end{bmatrix}$$

- Veza sa **A**:

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{P}^{-1} \cdot \dot{\mathbf{x}}(t) = \underbrace{\mathbf{P}^{-1} \cdot \mathbf{A} \cdot \mathbf{P}}_{\tilde{\mathbf{A}}} \cdot \tilde{\mathbf{x}}(t) + \underbrace{\mathbf{P}^{-1} \cdot \mathbf{B}}_{\tilde{\mathbf{B}}} \cdot \mathbf{u}(t)$$

daje

$$\mathbf{A} \cdot \mathbf{P} = \mathbf{P} \cdot \tilde{\mathbf{A}}$$

Svojstvene vrednosti matrice

- Rešenje jednačine

$$\mathbf{A}\cdot\mathbf{P}=\mathbf{P}\cdot\widetilde{\mathbf{A}}$$

tj.

$$\mathbf{A}\cdot\begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n\end{bmatrix}=\begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n\end{bmatrix}\cdot\begin{bmatrix}\lambda_1 & & & \mathbf{0}\\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n\end{bmatrix}$$

Svodi se na *n* relacija čijim rešavanjem ćemo dobiti **P**

$$\mathbf{A}\cdot\mathbf{p}_i=\mathbf{p}_i\lambda_i,\quad i=1,2,\ldots n$$

- Prethodna jednačina se može napisati kao

$$(\mathbf{A}-\lambda_i\mathbf{I})\cdot\mathbf{p}_i=0,\quad i=1,2,\ldots n$$

i njeno netrivijalno rešenje je za

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

λ_i su svojstvene vrednosti matrice **A**

$\mathbf{p}_i$ su svojstveni vektori matrice **A**

Računanje E i F

- Kada nađemo svojstvene vrednosti i svojstvene vektore matrice A, tada možemo izračunati E i F

$$\mathbf{A} = \mathbf{P} \cdot \widetilde{\mathbf{A}} \cdot \mathbf{P}^{-1}$$

- Računanje E i F:

$$\mathbf{E} = \mathbf{\Phi}(T) = \mathbf{P} \cdot \widetilde{\mathbf{\Phi}}(T) \cdot \mathbf{P}^{-1}$$

$$\mathbf{F} = \left(\int_0^T \mathbf{\Phi}(t) dt \right) \cdot \mathbf{B} = \left(\int_0^T \mathbf{P} \cdot \widetilde{\mathbf{\Phi}}(t) \cdot \mathbf{P}^{-1} dt \right) \cdot \mathbf{B} = \mathbf{P} \cdot \left(\int_0^T \widetilde{\mathbf{\Phi}}(t) dt \right) \cdot \mathbf{P}^{-1} \cdot \mathbf{B} = \mathbf{P} \cdot \widetilde{\mathbf{A}}^{-1} \cdot \left(e^{\widetilde{\mathbf{A}}T} - \mathbf{I} \right) \cdot \mathbf{P}^{-1} \cdot \mathbf{B}$$

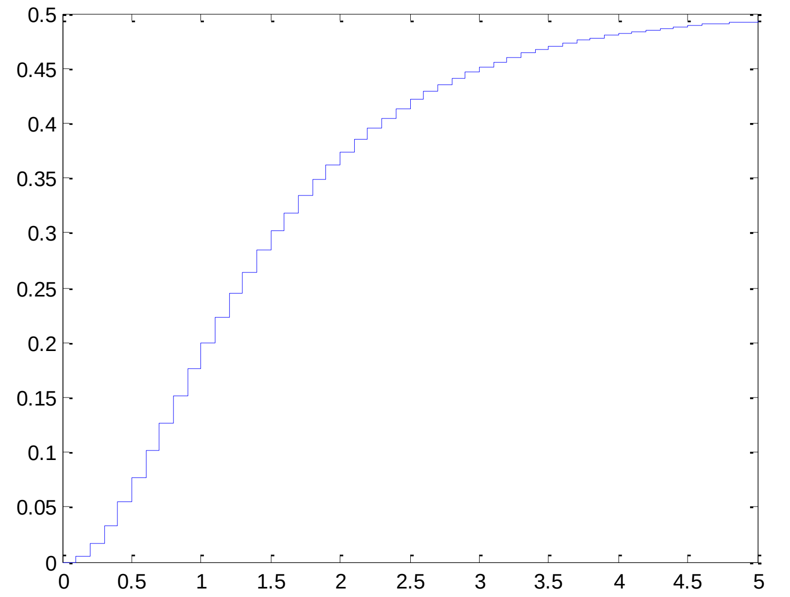
Primer 1 (1)

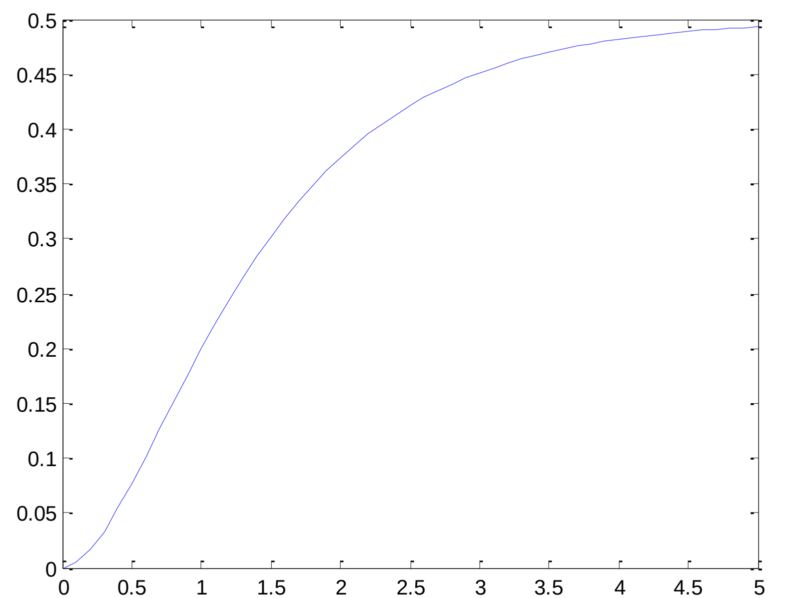
```
>> A = [-3  -2; 1   0];
>> B = [1; 0];
>> C = [0  1];
>> D = 0;

>> [P,lambda]=eig(A)
P =
   -0.8944    0.7071
    0.4472   -0.7071
lambda =
    -2     0
     0    -1

>> Ts = 0.1;          % perioda odabiranja
>> E = P*expm(lambda*Ts)/P
E =
    0.7326   -0.1722
    0.0861    0.9909
>> F = P/lambda*(expm(lambda*Ts)-eye(2))/P * B
F =
    0.0861
    0.0045

>> u = ones(1,5/Ts+1);
>> x(:,1) = [0;0];    % =x0
>> for k=1:length(u)-1
       x(:,k+1) = E*x(:,k) + F*u(k);
   end
>> y = C*x + D*u;
>> t = (0:length(y)-1)*Ts;
>> stairs(t,y)        % crtanje sa zadrškom
>> plot(t,y)          % … kao da je kontinualan
```





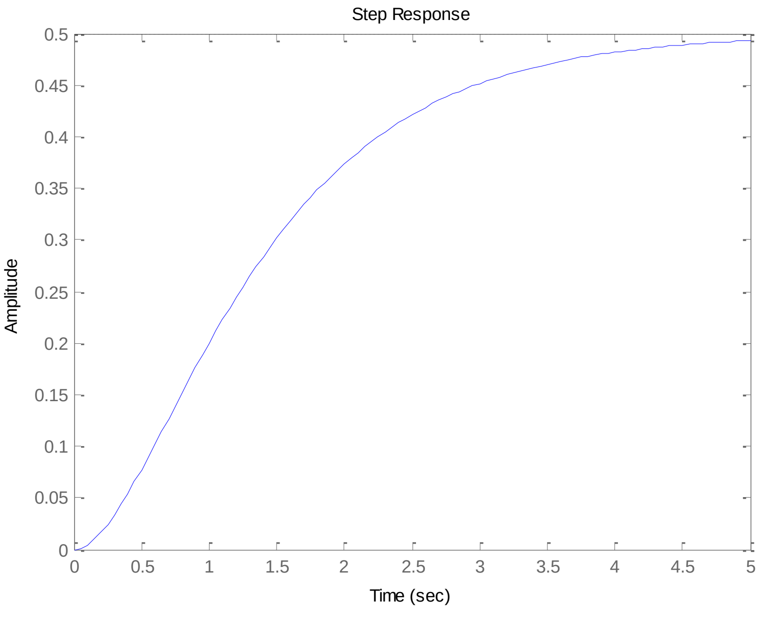
Primer 1 (2)

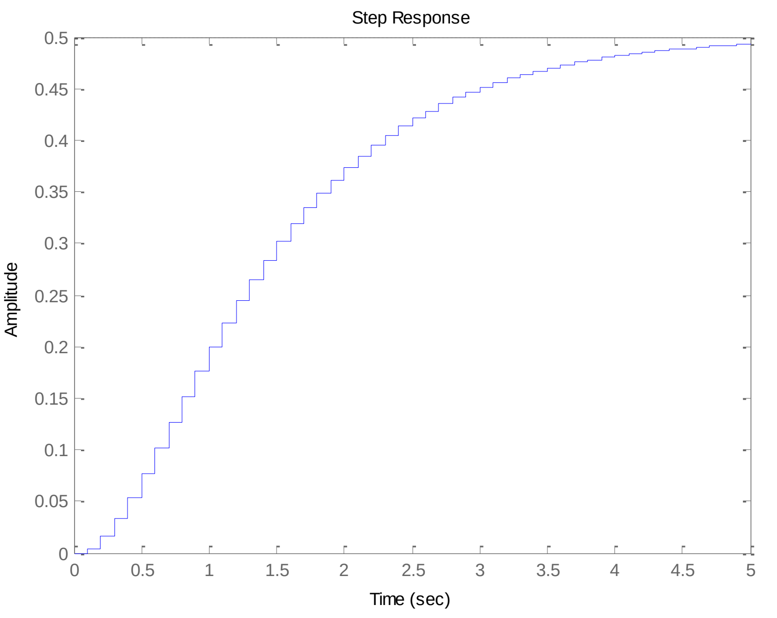
```
>> A = P*lambda/P  % provera
A =
    -3    -2
     1     0

>> m=ss(A,B,C,D);     % kont.model
>> md=c2d(m,Ts)       % diskretizacija
a =
            x1        x2
   x1   0.7326   -0.1722
   x2  0.08611    0.9909
b =
             u1
   x1   0.08611
   x2  0.004528
c =
       x1  x2
   y1   0   1
d =
       u1
   y1   0
Sampling time: 0.1
Discrete-time model.

>> expm(A*Ts)         % E=fundamentalna matrica
ans =
    0.7326   -0.1722
    0.0861    0.9909

>> step(m,5)          % odziv kont.modela
>> step(md,5)         % odziv diskretnog modela
```





Diskretna funkcija prenosa

- Diskretna funkcija prenosa je odnos kompleksnih likova vremenski diskretizovanih signala izlaza i ulaza

$$\frac{Y(z)}{U(z)} = W(z)$$

Diskretna funkcija prenosa W(z)

(za nulto poč. stanje)

$$U(z) \longrightarrow \boxed{W(z)} \longrightarrow Y(z)$$

- Diskretna funkcija prenosa se realizuje kao količnik polinoma po z

$$W(z) = \frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_2 z^2 + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0}$$

ili

$$W(z^{-1}) = z^{-(n-m)} \frac{b_m + b_{m-1} z^{-1} + \cdots + b_1 z^{-(m-1)} + b_0 z^{-m}}{1 + a_{n-1} z^{-1} + \cdots + a_1 z^{-(n-1)} + a_0 z^{-n}}$$

Diferencna jednačina

$$Y(z) = W(z)U(z) = \frac{B(z)}{A(z)}U(z) \;\Rightarrow\; A(z)Y(z) = B(z)U(z)$$

- U razvijenom obliku (uz: $d = n - m$)

$$\left(1 + a_1 z^{-1} + \cdots + a_{n-1} z^{-(n-1)} + a_n z^{-n}\right)Y(z)$$
$$= z^{-d}\left(b_0 + b_1 z^{-1} + \cdots + b_{m-1} z^{-(m-1)} + b_m z^{-m}\right)U(z)$$

$$Y(z) + a_1 z^{-1} Y(z) + \cdots + a_n z^{-n} Y(z) = b_0 z^{-d} U(z) + \cdots + b_m z^{-d-m} U(z)$$

- Na osnovu osobine "pomeranje u vremenskom domenu"

$$y(k) + a_1 y(k-1) + \cdots + a_n y(k-n)$$
$$= b_0 u(k-d) + \cdots + b_m u(k-d-m)$$

$$y(k) = -\sum_{i=0}^{n} a_i y(k-i) + \sum_{i=0}^{m} b_i u(k-d-i)\,, \qquad k \equiv kT, k = 0,1,\ldots$$

Diferencna jednačina

Diskretna funkcija prenosa

- Primenimo Laplasovu transformaciju na vremenski diskretan model u prostoru stanja sa jednim ulazom i jednim izlazom

$$\mathbf{x}(k+1) = \mathbf{E}\cdot\mathbf{x}(k) + \mathbf{F}\cdot u(k) \quad / L$$

$$\mathbf{X}(s)e^{sT} = \mathbf{E}\cdot\mathbf{X}(s) + \mathbf{F}\cdot U(s)$$

- Ranije uvedena smena $z = e^{sT}$

$$z\cdot\mathbf{X}(z) = \mathbf{E}\cdot\mathbf{X}(z) + \mathbf{F}\cdot U(z)$$

$$(z\mathbf{I}-\mathbf{E})\cdot\mathbf{X}(z) = \mathbf{F}\cdot U(z)$$

$$\mathbf{X}(z) = (z\mathbf{I}-\mathbf{E})^{-1}\cdot\mathbf{F}\cdot U(z)$$

- Izlaz...

$$Y(z) = \mathbf{C}\cdot\mathbf{X}(z) + \mathbf{D}\cdot U(z)$$

$$Y(z) = \mathbf{C}\cdot(z\mathbf{I}-\mathbf{E})^{-1}\cdot\mathbf{F}\cdot U(z) + \mathbf{D}\cdot U(z)$$

$$\frac{Y(z)}{U(z)} = W(z) = \mathbf{C}\cdot(z\mathbf{I}-\mathbf{E})^{-1}\cdot\mathbf{F} + \mathbf{D}$$

…

Matrice vremenski diskretnog modela su:

$$\boldsymbol{E} = \begin{bmatrix} 0.7326 & -0.1722 \\ 0.0861 & 0.9909 \end{bmatrix}, \qquad \boldsymbol{F} = \begin{bmatrix} 0.0861 \\ 0.0045 \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} 0 & 5 \end{bmatrix}, \qquad \boldsymbol{D} = 0$$

Diskretna funkcija prenosa

$$\frac{Y(z)}{U(z)} = W(\mathrm{z}) = \boldsymbol{C}(\mathrm{z}\boldsymbol{I} - \boldsymbol{E})^{-1}\boldsymbol{F} + \boldsymbol{D}$$

$$W(\mathrm{z}) = \begin{bmatrix} 0 & 5 \end{bmatrix}\left(z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.7326 & -0.1722 \\ 0.0861 & 0.9909 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0.0861 \\ 0.0045 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 0 & 5 \end{bmatrix}\begin{bmatrix} z - 0.7326 & 0.1722 \\ -0.0861 & z - 0.9909 \end{bmatrix}^{-1}\begin{bmatrix} 0.0861 \\ 0.0045 \end{bmatrix} = \frac{0.02264z + 0.02049}{z^2 - 1.724z + 0.7408}$$

$$= \frac{0.02264z^{-1} + 0.02049z^{-2}}{1 - 1.724z^{-1} + 0.7408z^{-2}}$$

Diferencna jednačina

$$(1 - 1.724z^{-1} + 0.7408z^{-2})Y(z) = (0.02264z^{-1} + 0.02049z^{-2})U(z)$$

$$y(k) - 1.724y(k-1) + 0.7408y(k-2) = 0.02264u(k-1) + 0.02049u(k-2)$$